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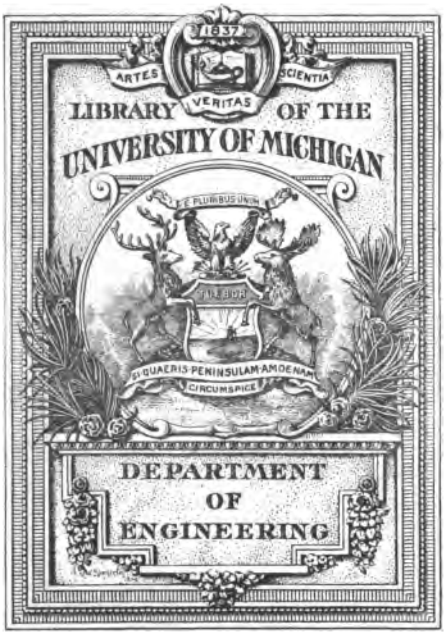

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AN
INTERMEDIATE COURSE OF
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AN INTERMEDIATE COURSE OF MECHANICS

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PREFACE

THIS book has been written as an accompaniment to a First Year Course of College Lectures.

It has not been written from the point of view of Examinations. It, however, includes those portions of Mechanics required for the Intermediate Pass Examination of the University of London.

Stress has been laid on fundamental principles rather than on fulness of detail; the latter is apt to overwhelm a Student who is just beginning a subject.

I desire to express my thanks here to Mr. N. EUMORFOPOULOS, B.Sc., Fellow of the College, who has assisted me with the examples.

A. W. P.

UNIVERSITY COLLEGE,
LONDON.
June, 1905.

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INTERMEDIATE MECHANICS

CHAPTER I

INTRODUCTION

Of all the phenomena of Nature the most direct and most universal are concerned with the *motions* of bodies. Stones fall, rivers flow, the stars circle in the sky. We ourselves move from place to place, and set other things in motion or keep them at rest.

What are the conditions under which motion takes place? We can give various more or less vague answers to this question. It is the business of the Science of Mechanics to find out what these conditions are, and express them in definite (*i.e.* scientific) language. The language that is employed is to a large extent the language of Mathematics. Mechanics is therefore often regarded as a branch of Mathematics, and is called Applied Mathematics. But the phenomena which are described are indeed merely the most universal of the phenomena of Nature common to both animate and inanimate things; and the properties of these, when examined in their non-vital aspects, form the Science of Physics, of which, therefore, Mechanics must be considered as a part.

Kinematics. It is convenient to divide this science into two sections. We may answer the question, What is the best way of distinguishing one example of motion from any other, and of describing the difference? The branch in which this question is answered is called Kinematics (*κινέω*, I move).

Dynamics. Or we may ask the question, How can a definite motion be produced or prevented from taking place? The answer to this question is classed under the head of Dynamics.

It is clear that we must know how to describe motion before we can give a satisfactory answer to this second question. We begin, therefore, with Kinematics.

Space and Time. A child soon learns what is meant by *here* and *there*. It can reach its toy, or perhaps it is out of reach. It has to *wait*, and thus acquires a knowledge of *time*. In scientific language, everything occupies Position in Space and in Time. It is not necessary to inquire here as to what Space and Time really are. This is a very difficult question, to which no certain answer has yet been given. It is sufficient for our purpose to accept the popular conceptions in regard to them. The properties of Space are discussed in Geometry; these will be assumed to be known. The following definitions however must be recalled:

The length of a straight line is the distance or amount of space between its two ends.

Two straight lines are of equal length if when one is placed (or supposed to be placed) in coincidence with the second, their ends also coincide. Any number of straight lines may in this way be tested as to their equality to one another.

The length of a straight line is stated to be an integral multiple of that of another if a number of straight lines equal to the latter and placed end to end along the former are exactly included in it. The smaller is stated to be a fractional part of the larger, the fraction being unity divided by a whole number.

In practice the lines are compared by means of a *scale* or *measure.* This is a bar of wood, metal, or other substance, and either two marks are made on it, one near each end, or the ends themselves of the bar are used as points of reference. Such a bar represents a moveable line which can be superposed upon the line under comparison. Similar scales are then placed in succession along the line as described, or more usually the same scale is placed successively with its first reference point moved to where its second reference point was previously, and the number of such steps counted. Such a scale is called a standard or unit of length.

In general, the length of one straight line will not be an integral multiple of that of the unit. It is found, *e.g.*, that it is greater than m units and less than $m + 1$. The excess over m units is then compared with a smaller unit. If the excess is not an integral multiple of the smaller unit, the fresh excess can be compared with a still smaller unit, and so on. The length of the whole line is the sum of the lengths of its parts. The whole length can, therefore, be specified approximately in terms of this graded series of units; and as the excess at each stage is smaller than that at any preceding stage, the error committed by neglecting to take the final excess into account can theoretically be made as small as we please by a continuation of the process.

The properties required in order that any such standard as that referred to should be perfect are: (i) Permanence. (ii) Convenience.

By *permanence* we mean that the distance between the reference points must remain invariable. No substance is known to fulfil this requirement in perfection. All substances change their size with change in temperature. Care has therefore to be taken to use the standard at a definite temperature, or else to know and correct for its changes in length. Invar (an alloy of steel and nickel) undergoes least variation with change in temperature, and is therefore now frequently used for such standards. Every standard requires to be very carefully preserved; copies of it—which can be compared with it from time to time—are made for actual use.

Convenience determines the length that shall be selected; it necessitates that there shall be different units for different purposes. The chief units that have been legalised are the following:

1. **The Standard Yard.** This is the British standard, and is defined as the distance, measured at 62° F., between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the office of the Exchequer. The fractional parts selected as lesser units are a one-third part, called a *foot*, and a one-twelfth part of a foot, called an *inch*. The inch is then variously subdivided into eight, ten, or more equal parts, to which no special names are given.

For long lengths multiples of the yard are taken as units; thus 1,760 yards equal one *mile*. Other multiples are also sometimes used, but do not require enumeration in this place (see Tables at end).

2. **The International Metre.** The British units described above do not lend themselves readily to purposes of calculation. Our numbers are based upon the system of ten, and much greater convenience is obtained by making the relation between any two units a multiple of ten. This rule is followed in the *International System*, which will probably supersede all National units. This unit is the outcome of the labours of a committee appointed during the French Revolution (1795).

The unit of length adopted was intended to be the ten-millionth part of a quarter meridian of the Earth, and is called a *metre.* The present definition of it is the distance at the melting point of ice between the centres of two lines engraved upon the polished surface of a platinum-iridium bar of a nearly X-shaped section, called the International Prototype Metre. This is kept at the International Bureau of Weights and Measures at Sèvres, near Paris, and copies of it are supplied to the different Governments. One-tenth of this unit is called a *decimetre*; one-hundredth, a *centimetre*; one-thousandth, a *millimetre.* One thousand metres equal one *kilometre.*

In scientific work this standard has already practically superseded all others; in British commerce it is only an alternative legal standard. The yard or its fractional parts, like any other lengths, can be compared with the metre. One inch is very nearly indeed equal to 2·54 centimetres.

Measurement of Time. Time is another physical magnitude capable of being measured; and exactly as in the case of space, lengths of time can only be compared with one another by finding out how many times some chosen standard length of time is included in each. Thus we may express how many times the sun rose while a certain event proceeded. The length of time between two sunrises is here the unit employed.

This unit, however, is found not to be constant; it can be compared to a scale made out of india-rubber: there would be no consistency amongst measurements made with such a scale. And intervals of time which we have reason to believe are the same appear not to be the same when compared with the time between two sunrises.

The length of time which we have reason to believe is so nearly constant as to serve as a unit is that taken for the earth to turn round once on its axis. This is called a *sidereal day* because it is the time which a star takes in apparently going once round the earth. The *mean value* of the time between two passages of the sun across the meridian is compared with this, and is called a *mean solar day.* This day is divided into twenty-four equal parts, called hours, each hour into sixty equal *minutes*, and each minute into sixty *mean solar seconds.* Thus, by definition, there are 24 × 60 × 60, that is 86,400, mean solar seconds in a mean solar day, and by observation it is found that there are 86,164 mean solar seconds in a sidereal day. The division of the whole day into equal parts is effected experimentally by means of a *clock.* Each swing of the pendulum of a clock, which has been compensated so as to be unaffected by temperature, occupies a constant time. The mechanism of a clock consists partly in a spring or weights arranged to maintain the pendulum in motion, and partly of a counting apparatus to record the number of vibrations that take place.[1]

The reader may wonder how we decide that a platinum scale or time of rotation of the earth remains invariable. The reason for asserting their *practical* invariability is that measurements made on the assumption of their invariability are found to be on the whole more consistent with one another than if the contrary supposition is made; and, moreover, the laws of mechanics deduced from experiments take a simpler form than they otherwise would do. There is no other criterion of invariability.

In reality there are astronomical observations which can be most simply expressed by supposing that *the sidereal day is gradually lengthening*; the increase is, however, so exceedingly slight that it is only during astronomical periods that it can become perceptible. No account need be taken of it in other departments of science or in civil life.

We can assert that when a body moves, any one point in it travels along a *continuous* path, joining its initial and final positions. This is equivalent to saying that it is never annihilated or re-created. Even if the body becomes invisible, as when it is converted into a gas, evidence of its continuous existence is so usually forthcoming that the above assertion is made with confidence. The path may be either straight or curved. If different characteristic points of a body are examined, their paths in general differ from one another in shape, and may intersect one another. By far the simplest case is that in which all the paths are parallel straight lines; the motion is then called *Rectilinear Translation.* Throughout the next five chapters the motion will be supposed to be of this simple kind.

---

[1] It should be noticed that in measuring time the actual observation is a space measurement, *viz.* the movement of a star or of the hand of the clock. When we say, therefore, that a motion lasts for a certain time, we in reality merely compare the given motion with the concomitant motion of the star or clock hand (compare Mach, *Science of Mechanics*, p. 223).

CHAPTER II

RECTILINEAR TRANSLATION OF A RIGID BODY

A *RIGID BODY* is one which preserves its size and shape unchanged; and although no body comes up to this definition, many satisfy it so nearly that they may be treated as rigid. For example, a cricket ball under ordinary circumstances is nearly rigid, a cannon ball still more so.

In the translation of a rigid body, all points move in the same way; we need therefore only refer to the motion of one of them. In rectilinear translation this point moves backwards or forwards along a straight line. The straight line connecting its position at any instant with its position at any succeeding instant is called its *Displacement* in the intervening time. Two positions are thus required to determine the displacement. If its displacements in successive intervals are all in a forward sense along the line (*e.g.* from left to right), its displacement in the total interval is the arithmetic sum of the several displacements; for

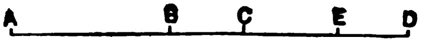


Fig. 1.

in Fig. 1, AD = AB + BC + CD.

In the next interval let the point move backward to E; the total displacement is the line AE. This is the *algebraic* sum of the separate displacements, provided that we give the backward one a negative sign. The total displacement is called the *resultant*; the separate ones, *components*. If the point returns to its initial position, the total displacement is zero, whatever position it may have occupied in the interval.

It has been said that every event occupies time. The displacement divided by the time is the *average velocity*: it is the average value of the displacement in unit time. Thus, if in ten hours the displacement is thirty miles, the quotient is three, and the average velocity is three miles per hour. If the quotient is the same, whatever interval be examined, the average velocity is constant: it is then said to be *the* velocity; it expresses the rate of going at each instant. In this case we can write $s = vt$, where s = displacement, v velocity, and t the time; that is to say, displacement and time are proportional to one another, and v is the factor of proportionality.

Where the quotient depends upon the interval considered, the rate of motion at any instant is not obtained so easily. Consider shorter and shorter intervals, each containing the instant in question. Let the displacements in these intervals be measured and the quotients calculated. These quotients as a rule will have various values; but *in every actual case* the values will at last tend to become more and more nearly the same as the interval approaches zero length. This limiting value is called the velocity at the instant in question.

In all such cases the relation between displacement and time is more complicated than when the velocity is a constant. In illustration of the process indicated above, let us suppose that the formula connecting the displacement and the time interval is $s = kt^2$, where k is a constant. In this case the displacement increases much faster than the time; it is very nearly the law followed by a falling stone. At the instant t_1 let the displacement be s_1, and at t_2 let it be s_2; then

$$s_1 = kt_1^2$$
$$s_2 = kt_2^2,$$

whence $s_2 - s_1 = k(t_2^2 - t_1^2) = k(t_2 + t_1)(t_2 - t_1)$.

Now $s_2 - s_1$ is the displacement during the time interval $t_2 - t_1$, and consequently the *average* velocity ($\bar{v}$) in this interval is

$$\bar{v} = \frac{s_2 - s_1}{t_2 - t_1} = k(t_2 + t_1);$$

$\bar{v}$ therefore depends upon both t_2 and t_1.

If now we take the interval as short as we please, t_2 becomes more and more nearly equal to t_1, and therefore $t_2 + t_1$ approaches the value $2t$, where t is an instant between them. The velocity v at the time t is therefore given by

$$v = 2kt.$$

This result can be stated in words as follows: If the displacement is proportional to the square of the time, the velocity bears *twice* the ratio to the time itself. Example: if $s = 16t^2$, it follows that $v = 32t$.

Similarly, if $s = kt^3$,

$$s - s_1 = k(t_2^3 - t_1^3) = k(t_2^2 + t_1t_2 + t_1^2)(t_2 - t_1),$$

and

$$\bar{v} = \frac{s_2 - s_1}{t_2 - t_1} = k(t_2^2 + t_1t_2 + t_1^2).$$

As t_1 and t_2 coalesce to an intervening instant t, this becomes $v = k(t^2 + tt + t^2) = 3kt^2$; or in words, whenever the displacement is proportional to the cube of the time the velocity bears *three times* the ratio to the square of the time.

A simple equation connecting the displacement with the time cannot always be found. In such cases we must resort to a diagrammatic way of dealing with the problem.

In this book it is only simple cases that we are concerned with; but the diagrammatic method helps in explaining these also, and consequently will now be given.

Draw two lines at right angles to one another; these are called *axes of reference*, and their point of intersection (O) is called *the origin* (Fig. 2).

Let lengths measured from O be taken along OS to represent displacements on any suitable scale, and along OT to represent time intervals. Erect verticals at various positions, and cut off lengths of these representing (on the vertical scale) the displacements at the times corresponding to the points where they cut the horizontal axis. Thus, travelling in a railway train, the instant of passing each quarter-mile post may be observed. The verticals would then be drawn, cutting OT at the points corresponding to these times, and the lengths cut off would each be greater than the preceding one by the length representing a quarter of a mile.

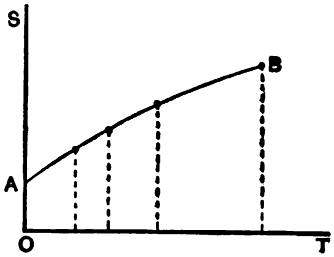


Fig. 2.

Now the displacement never changes abruptly, otherwise a body could be in two places at the same time. Hence we know that if the displacement at all intermediate times had been represented in the above way, the points determined would form a *continuous* curve. If thus the measurements are made at times sufficiently close together, and we draw a *smooth* curve, such as AB, passing through the determined points, the vertical distance of any point of this curve from OT gives the displacement at the corresponding instant. Such a curve is called a *displacement curve.* The reader must be warned *that this curve is not the path of motion*; the diagram is a purely conventional way of indicating the connection between the displacement and time instead of by an algebraic equation. The two quantities s and t are said to be *plotted* one against the other. When the length which is taken to represent unit displacement is the same as that taken to represent unit time, the two quantities are said to be *plotted on the same scale.*

The same kind of diagram can be used to show the connection between any two quantities whatever, and is frequently employed for this purpose.

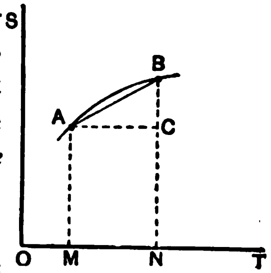


Fig. 3.

What property of such a curve represents the *velocity*? To answer this question recourse must be had to the definition of average velocity and comparison be then made with the diagram.

In Fig. 3, where $\overline{AB}$ is the displacement curve, $\overline{BC}$ is the displacement taking place in the interval of time $\overline{MN}$ or $\overline{AC}$; hence the average velocity $\bar{v}$ in this interval is numerically equal to the length BC divided by the length AC, provided that s and t are plotted on the same scale.

This ratio is equal to the tangent of the angle BAC, where BA is the straight line joining A and B. If on the other hand unit length in the diagram represents m units of displacement and n units of time,

$$\bar{v} = \frac{s}{t} = \frac{m.\overline{\mathrm{BC}}}{n.\overline{\mathrm{AC}}} = \frac{m}{n} \tan \widehat{\mathrm{BAC}}.$$

We shall call $\tan \widehat{\mathrm{BAC}}$ the *slope* of the line $\overline{\mathrm{AB}}$.

Now the velocity is the limiting value which $\bar{v}$ takes when M and N approach indefinitely near to one another; but A and B also then draw near to one another; the velocity is therefore m/n times the limiting value of the slope of AB as B comes into contact with C. But provided A and B are so close together that the curve between them is either wholly convex or wholly concave toward either axis, the slope of the straight line $\overline{\mathrm{AB}}$ is intermediate between the slopes of the geometric tangents at A and B, and therefore coincides with them when they coincide. Hence the velocity at the time M is equal to m/n times the slope of the geometric tangent at A.

It follows that if the curve were anywhere perpendicular to OT, the velocity would then be infinite. No infinite velocity is known to occur; very large velocities are known, however, and with these the curve will be nearly vertical unless the scale-ratio m/n is chosen big.

In actual cases—as in the shunting of a train, for example—the displacement curve may be very irregular. Fig. 4 illustrates such a case. The average velocity has very fluctuating values as the interval MN varies. There would be no sign of approach to a limiting value for the velocity until B is so near A that all further variation shall be in one sense; hence the restriction that the curve between A and B shall be wholly convex or wholly concave.

We regarded a displacement as negative when taking place in a backward sense; in such a case the velocity also will be taken as negative.

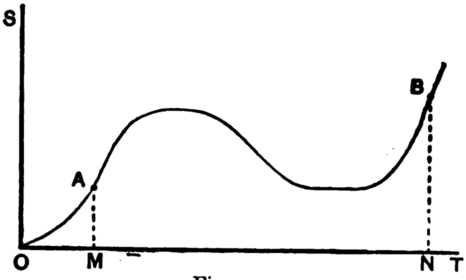


Fig. 4.

Unit of Velocity. Since velocity is a length divided by a time, it can neither be measured with reference to space alone nor to time alone. Now if we divide one length by another, it is clear that the quotient gives the number of times the smaller is included in the larger. But it is by no means obvious what the meaning of dividing the numerical value of a length by that of a time will be, because the quantities are not of the same kind. Without trying to give an ultimate answer to the question as to what velocity really is, we take the numerical value obtained as giving the ratio the velocity bears to a certain velocity (*i.e.* a magnitude of the same kind) taken as the standard unit. Now in the equation $s = vt$, v is obviously unity if s and t are each unity. Hence we define unit velocity as that which a body has when it travels unit distance in unit time at a constant rate. The unit distance and time may be any of those previously defined; thus we may take as unit a velocity of 1 ft. per second or 1 ft. per hour or 1 cm. per second. The first and last are very commonly chosen. The unit of velocity is known as a *derived* unit, while those of length and time upon which it is based are *fundamental* units.

Change of Unit. The same rate may, of course be compared with any one of the units. Knowing its value expressed in terms of one unit, it may be required to know its value in terms of any other. In finding this new value we are said to transform from one system to another. The change may be effected in detail, as in the following example:

Example: A velocity is known to be 12 cm. per second: required, its value if the fundamental units are the foot and hour.

$$12 \text{ cm.} = \frac{12}{2{\cdot}54 \times 12} \text{ ft.}$$

$$1 \text{ second} = \frac{1}{3{,}600} \text{ hour.}$$

Thus travelling 12 cm. in one second is the same rate as travelling $\frac{1}{2{\cdot}54}$ ft. in $\frac{1}{3{,}600}$ hour,

or $\frac{3{,}600}{2{\cdot}54}$ ft. in one hour;

i.e. 141·7 ft. in one hour.

The required answer is therefore 141·7 ft. per hour.

This process can be abbreviated as follows:

Let v ft. per hour be the equivalent value. Remembering that the time enters in the denominator, write

$$v\,\frac{\text{ft.}}{\text{hour}} = 12\,\frac{\text{cm.}}{\text{sec.}};$$

$$\text{or}\quad v\,\frac{12 \times 2{\cdot}54\ \text{cm.}}{3{,}600\ \text{sec.}} = 12\,\frac{\text{cm.}}{\text{sec.}};$$

$$\text{or}\quad \frac{v \times 2{\cdot}54}{3{,}600} = 1;$$

$$\text{or}\quad v = \frac{3{,}600}{2{\cdot}54} = 141{\cdot}7.$$

Hence the answer required is $141{\cdot}7\,\frac{\text{ft.}}{\text{hour}}$.

In the above the name of the unit is treated as though it were an algebraic quantity. The abbreviation in practice is carried still further by simply writing in equivalents and cancelling, thus:

$$v\,\frac{\overset{30{\cdot}48\,\cancel{\text{cm.}}}{\cancel{\text{ft.}}}}{\underset{3600\,\cancel{\text{sec.}}}{\cancel{\text{hour}}}} = 12\,\frac{\cancel{\text{cm.}}}{\cancel{\text{sec.}}}$$

$$\therefore\quad v = \frac{12 \times 3600}{30{\cdot}48} = 141{\cdot}7$$

Acceleration. When a velocity is not constant its rate of increase is called *acceleration.* The average value of this is obtained by dividing the total increase in velocity by the time interval. Thus, if the velocity is v_1 at t_1 and v_2 at t_2, the average acceleration

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1}.$$

Exactly the same considerations apply to this ratio as to velocity itself. The acceleration at any instant is obtained by finding the limiting value of this ratio, when $t_2 - t_1$ is indefinitely small. A *velocity diagram* can be obtained by plotting v against t, and if m/n be the scale-ratio, the acceleration is m/n times the slope of the geometric tangent to the curve.

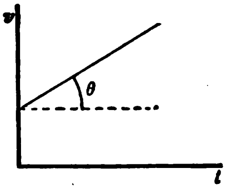


Fig. 5.

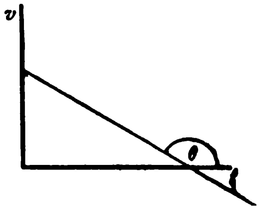


Fig. 6.

For the present only cases in which the acceleration is constant will be considered. The velocity curve is then a straight line, sloping upwards as in Fig. 5 if the velocity is increasing with time, or downwards as in Fig. 6, if it decreases. In each case the acceleration is

$\frac{m}{n} \times \tan \theta$; when θ is $> 90°$ and $< 180°$ both $\tan \theta$ and a are negative. This is consistent with the defining equation, for in the latter case $v_2 < v_1$ if $t_2 > t_1$.

Unit of Acceleration. Since, when constant, the acceleration is the increase of velocity in unit time, the time is involved twice over. The unit of acceleration may be that under which a velocity of 1 ft. per second is gained per second, or 1 cm. per second is gained per second: these may be abbreviated into $1 \frac{\text{ft.}}{\text{sec. sec.}}$ or $1 \frac{\text{ft.}}{\text{sec.}^2}$ and $1 \frac{\text{cm.}}{\text{sec. sec.}}$ or $1 \frac{\text{cm.}}{\text{sec.}^2}$ respectively; and if the numerical value of the acceleration is a, it means that it is a times the unit employed. A derived unit of acceleration may be obtained from any other of the fundamental units of length or time; *e.g.* we may speak of $1 \frac{\text{mile}}{\text{day}^2}$, which means that the velocity increases 1 mile per day in each day.

Change of Units. The change from one unit to another is effected in the same type of way in every case, and the method is sufficiently illustrated by the following example:

Example: An acceleration is equal to $10 \frac{\text{cm.}}{\text{sec.}^2}$.

what is its numerical value when feet and minutes are the fundamental units?

$$\text{Let} \qquad y\,\frac{\text{ft.}}{\text{min.}^2} = 10\,\frac{\text{cm.}}{\text{sec.}^2};$$

$$\text{then} \qquad y\,\frac{30{\cdot}48 \text{ cm.}}{(60 \text{ sec.})^2} = 10\,\frac{\text{cm.}}{\text{sec.}^2};$$

$$\text{or} \qquad y = \frac{60^2 \times 10}{30{\cdot}48}$$

$$= 1{,}181.$$

I.e. the acceleration is equivalent to 1,181 $\frac{\text{ft.}}{\text{min}^2}$.

Calculation of Displacement. The calculation of the displacement when the velocity is constant is effected at once from the equation $s = vt$. When the velocity is variable, no such simple rule can be given, and a special formula is needed for each special mode of variation. We have seen that where the displacement is proportional to the square of the time, the velocity bears double the ratio to the time itself. But if the velocity is proportional to the time, the increase in unit time is constant, *i.e.* the acceleration is constant. Hence we have the following converse proposition :—

When the acceleration is constant, and the velocity is zero when $t = o$, the displacement is equal to half the acceleration multiplied into the time squared, or in symbols:

$$\text{If} \qquad v = at,$$

$$\text{then} \qquad s = \tfrac{1}{2}at^2,$$

If the acceleration is constant, but the velocity is not zero when $t = o$, we must write $v = v_0 + at$, where v_0 is the initial velocity. If the velocity had been v_0 throughout, the displacement would have been v_0t. Owing alone to the *increase* in velocity the displacement is $\frac{1}{2}at^2$; the total displacement due to both is $s = v_0t + \frac{1}{2}at^2$.

Note: This should be verified by assuming the last equation and calculating the limiting value of $(s_2 - s_1)/(t_2 - t_1)$, when $t_2 - t_1$ vanishes.

The displacement may also be found from the velocity diagram. When the velocity is constant the curve is a horizontal straight line (Fig. 7). Since the displacement is vt, it is proportional to the area ABNM. It is equal to it if v and t are plotted on the same scale and if the unit of area is the square formed from sides representing unit value of both v and t; if they are not on the same scale, the geometric area must be multiplied by the displacement corresponding to such unit area. Thus, if 1 in. vertically represents a velocity m, and horizontally represents a time n, the displacement corresponding to 1 sq. in. in the diagram is equal to mn. We shall call such an equivalent the *physical equivalent of unit area.* The displacement is mn times the area ABNM.

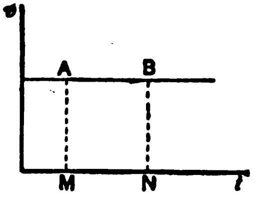


Fig. 7.

If the velocity is negative, the straight line representing it and the corresponding area are both below the time-axis. Since the displacement is negative, we will consider such an area to be negative also.

Now let there be a constant acceleration, so that the velocity curve is a sloping straight line (Fig. 8). The displacement in any short time (MN) is greater than if the velocity remained constant and the same as at the beginning of the interval, but less than if it remained the same as at the end of the interval; *i.e.* it is greater than the equivalent of the area *ac*NM, and less than that of the area *db*NM, and these only differ by the small rectangle *dbca*. If the whole interval be divided into short portions similar to MN, the total displacement must be greater than if the velocity in each interval remained the same as at the beginning of the same interval, and less than if throughout each interval it had the actual value at the end of it. But the total displacements on these two suppositions will only differ by the sum of such small rectangles as *dbca*, and

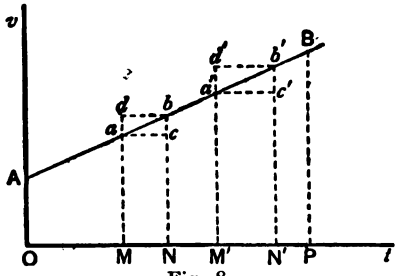


Fig. 8.

this sum is less than the rectangle formed by a line equal to the sum of their heights and a line equal to the width of the widest of them. Since, if the number be taken indefinitely great, the width of the widest can be made indefinitely small, the two suppositions lead to results differing from one another by an amount which can be made as little as we please by making the number of intervals as large as we please; and the actual displacement then differs indefinitely little from either of them, for it lies between the two coalescing values. Now the area ABPO is also intermediate to the two coalescing areas; and consequently this represents the actual displacement in the time OP. Since this figure is a trapezium, its area is $\frac{AO + BP}{2} \cdot OP$, or $\frac{v_0 + v_1}{2}t$, where v_1 is the final velocity.

But $v_1 = v_0 + at$,

$\therefore s = \frac{1}{2}(2v_0 + at)t = v_0t + \frac{1}{2}at^2$,

which is the result previously obtained.

Corollary. Since the displacement is always equal to the *mean velocity* multiplied into the time interval, the mean velocity in the case of constant acceleration is the *arithmetic* mean ol the velocities at the beginning and end of the interval.

Example: Let $a = -32$ ft. per sec. per sec. (N.B.:—This indicates a decreasing velocity), and

$$v_0 = 160 \text{ ft. per sec.}$$

The displacements and velocities at the end of integral numbers of seconds are here tabulated :

t in sec.......	1	2	3	4	5	6	7	8	9	10
s in feet.......	144	256	336	384	400	384	336	256	144	0
v in $\frac{\text{feet}}{\text{sec.}}$.......	128	96	64	32	0	−32	−64	−96	−128	−160

These are obtained from the formulæ:

$$s = 160t - 16t^2$$
$$\text{and } v = v_0 - 32t.$$

This is the case of a stone thrown upwards with initial velocity 160; its velocity decreases 32 ft. per sec. every second. This acceleration is said to be due to gravity, and is denoted by the letter g. The stone comes to rest in five seconds, and returns to its starting-point in ten seconds from the start. Each particular value of the displacement occurs twice—once on the upward and once on the downward journey, the times of occurrence being equidistant from start and finish respectively. That there must be two values of t for each value of s might be inferred from the fact that the equation is a quadratic in t. These times can be both positive only when the acceleration, as in this case, is negative.

Three variables, s, v, t, occur in these equations. The equation we have given for s connects it with t; an equation connecting it with v is obtained by eliminating t from the two equations

$$s = v_0t + \tfrac{1}{2}at$$
$$\text{and } v = v_0 + at$$

Whence

$$
\begin{aligned}
s &= v_0\left(\frac{v - v_0}{a}\right) + \tfrac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 \\
&= \frac{v_0 v - v_0^2}{a} + \tfrac{1}{2}\frac{v^2 + v_0^2 - 2vv_0}{a} \\
&= \frac{v^2 - v_0^2}{2a}
\end{aligned}
$$

or

$$v^2 = v_0^2 + 2as$$

Example: In the preceding problem this becomes

$$
\begin{aligned}
v^2 &= 160^2 - 64s \\
&= 64(400 - s)
\end{aligned}
$$

Thus, when $s = 144$

$$v^2 = 64 \times 256$$

and $v = \pm 8 \times 16 = \pm 128.$

Again, when $s = 400, v = 0.$

Change of Point of Reference. Two observations are required to determine a displacement, for the displacement is the line joining the point of departure with the point of arrival. The point of departure is called the *reference point.* Suppose now a passenger walks along a corridor train which is itself in motion he advances 10 ft., say, from the end of the carriage, and we may say that this is his displacement with respect to that end. But if measurements are made from the station, the increase of distance will depend in part upon the motion of the train. If the motion in the train and the motion of the train are both in the same sense, the displacement of the man with respect to the station is the sum of his displacement with respect to the train, and the displacement of the train with respect to its initial position; in any case it is the *algebraic* sum. This statement can be abbreviated thus:

$$P_E = P_T + T_E$$

where P stands for the passenger,
T for the train,
and E for a fixed point on the earth; the suffix in each case denoting the reference point.

Any other moving point may be taken as the reference point: in any particular case it will often be plain what point is taken without explicit mention of it; but *some* point is always referred to, implicitly or otherwise.

Example: Since, if a point A moves with respect to a point B, B necessarily moves *in the opposite sense* with respect to A, we may write

$$A_B = - B_A.$$

Bearing this in mind, we have the following additional equations connecting the positions of P, T, and E.

$$P_T = P_E + E_T$$
$$T_E = T_P + P_E$$

Since a velocity is numerically equal to a particular displacement, *viz.* that which would take place in unit time if the velocity were constant, the same remarks and rule apply to velocities. For a similar reason they apply also to accelerations. There is no meaning to be given to the phrases absolute displacement, absolute velocity, absolute acceleration; for all these quantities are measured with respect to some reference point, and their numerical values depend upon the particular reference point selected.

At the same time there may be physical reasons for selecting a particular reference point as being uniquely important. A point in the luminiferous ether regarded as a fixed body would appear to have claims as an ultimate point of reference, for it would be independent of any material body.

EXPERIMENT.

Let a small cylinder of brass (*e.g.* 3 cm. long and 2 cm. diameter) roll down an inclined plane, and observe the distance traversed in various intervals of time. Plot these distances against the time. Draw the geometric tangents to the curve at points corresponding to equal time intervals, and find their slopes (*i.e.* the trigonometric tangents of their inclinations to the time axis). These slopes measure the velocities at the corresponding times. They will be found to be approximately in arithmetic progression, indicating approximately constant acceleration.

If two cylinders of different size are taken, they will be found to have nearly the same acceleration. This is true whatever the size and whatever the material of the cylinders. The cylinders may conveniently be made of brass tubing filled with lead. An inclined plane is obtained by putting blocks under two of the legs of a table. If the table is not fairly smooth, a specially planed board must be laid on it. About 2 metres' length of path should be available. A centre seconds watch is needed.

EXAMPLES TO CHAPTER II.

1. A point travels 7 yds. in three minutes: find its velocity in feet per second.

2. A point travels 2 ft. in one minute, 3 ft. in the next two, and 5 ft. in the next three: calculate its average velocity in each interval and during the whole interval.

3. If the acceleration of a point is 10 ft. per sec. per sec., what is the velocity ten minutes after starting from rest?

4. If the acceleration is -10 cm. per sec. per sec., what is the velocity ten minutes after it is 4 cm. per sec.?

5. How far will a body go in one hour if its velocity is 12 cm. per sec.?

6. If the displacement of a body in feet is always ten times the square of the time in seconds, what is its velocity five seconds after starting? [If $s = kt^2$, $v = 2kt$.]

7. Express a velocity of 12 ft. per min. in miles per day and in centimetres per second.

8. Express an acceleration of 32·18 ft. per sec.^2 in centimetres per second^2.

9. Express a velocity of 60 miles per hour in feet per minute and in feet per second.

10. Calculate the displacements in the following cases:

Acceleration in each case − 32 ft. per sec.

(*a*) Initial velocity, 160 ft. per sec.; time, 2, 8, and 16 sec.

(β) Initial velocity, 10,000 ft. per min.; time, 3 min.

11. Calculate the displacement when the initial velocity is 16, the final velocity is 32, and the acceleration is 5.

12. What is the final velocity when the initial velocity is 3, the displacement in the interval 12, and the acceleration 4?

13. Obtain the velocity diagram for the case of a freely falling body; also for the case of one thrown vertically upwards with an initial velocity of 160 ft. per sec.

14. Two men a mile apart start simultaneously to walk toward one another with velocities of 260 and 180 ft. per min.: when and how far from their starting-points do they meet?

15. A train in passing from rest at one station to rest at another 5 miles away is first uniformly accelerated, and in the remainder of the journey its velocity decreases uniformly. If the whole journey takes nine minutes, find its maximum velocity. Show that it is the same whether the rates of increase and decrease of velocity are equal or otherwise.

16. A stone is let fall from a balloon which is rising with a velocity of 16 ft. per sec., and it strikes the ground twelve seconds later. What was its height above the ground when released?

17. If when at the same height it was *projected* with a velocity of 64 ft. per sec. downwards ***with respect to the balloon***, how long will it take to fall?

18. A body projected downwards describes 656 ft. in a certain interval and 1,168 ft. in the next equal interval: find this interval and the initial velocity.

19. A body projected upwards is at the same height after three seconds and after eight seconds from the start: what was its greatest height?

20. Two balls are projected vertically at the same instant, one upwards with velocity v_1, the other downwards with velocity v_2: when will they be equidistant from the point of projection?

CHAPTER III

RECTILINEAR DYNAMICS

In kinematics we restrict attention to one moving point or body, except in so far as another (and quite arbitrary one) is chosen as a reference point from which measurements are made. But in answering the question, How is motion produced? attention must be paid to other bodies as well, for it is the interaction between these and the body under particular examination which determines the motion of the latter. Thus, all bodies near the earth tend to drop toward the centre of the earth; they can only be prevented from doing so by interposing another body (*e.g.* a table). The earth, then, is a body which determines motion toward itself. Refined experiments show that this property is possessed by all bodies: two lumps of lead, for example, tend to move towards one another, though the motion produced is exceedingly slight compared with that produced by the earth in the same time. All the planets and the sun are found to control each other's motion to an amount depending upon their relative positions. The same is true for two balls fastened together by an elastic cord if the string is stretched and then released; but when it is unstretched, no motion is produced. Here, each part of the stretched string acts on adjacent parts, the final effect being the same as though the balls influenced each other directly. Electrified or magnetised bodies also tend to begin to move either away or toward one another. And we ourselves are constantly setting things into movement or stopping them. In order to describe all such phenomena in exact language, we begin with the following definitions:

When a velocity of a body alters in any way in consequence of the presence of another body, the second is said to exert *force* on the first.

The fact that the velocity does not change spontaneously—that is, without the presence of a second body—is otherwise expressed by saying that the body possesses *inertia* or *mass.*

The laws governing the action between bodies were only disentangled gradually (and even yet are certainly not fully known). The chief difficulty is in sufficiently isolating from others the two bodies concerned. But certain very general laws—apparently applicable to all changes of motion—were formulated in the seventeenth century (Kepler, Huygens, Newton), and are known as the Laws of Motion; these, stated in various forms, have formed the groundwork of all subsequent systems of mechanics.

We shall not adopt the historical method, but enumerate these principles in a modern form.

1. *All changes of motion are mutual, and take place in opposite senses.* By this is meant that no change of motion goes on without an accompanying change, in the opposite direction, of the motion of some other body.

Thus, when two billiard balls strike, one is accelerated forward, the other has its velocity in the same direction reduced. A shot is propelled forward; the gun which propels it recoils. A horse starts a cart; the cart acts as a drag upon the horse. When the cart is being stopped, it pushes the horse forward. The Earth determines the principal part of the motion of the Moon; the Moon, reciprocally, modifies the motion of the Earth.

2. *The accelerations of the two interacting bodies are not in general numerically equal to one another. Their masses are defined as being inversely proportional to the accelerations produced.*

The above definition makes the product of the mass into the acceleration the same for both, but of opposite sign. This product is taken as a measure of the magnitude of the force producing the given acceleration, and the direction of the acceleration is taken as the direction of the force. Hence the force on each due to the other is the same, but oppositely directed. This is known as the Principle of the Equality of Action and Reaction. Although it seems merely to follow from the definition, yet experiment is needed to prove that the definition of *mass* on which it is founded is a *useful* one. Thus it might be that if two masses were compared in terms of it, and then *each* compared with a third body, that the mass of the third would have a different value in each case. If this were so, it would be impossible to consider mass a definite property of a body. But the most precise experiments have failed to show any difference in the mass of a body in whatever way it has been ascertained.

Historically, the conception of force arose from our muscular sensations when giving motion to a body. The above mode of defining force is independent of these sensations, with which it can primarily have nothing to do; for there is no warrant for attributing sensations to inanimate objects when taking part in actions.

In so far as the sensation experienced by us can be estimated and compared with the force producing it, the two modes are not in disagreement. But we are only able to form rough judgments as to the intensity of a sensation; and consequently when accurate measures are concerned, it is impossible either to assert or deny that the sensations are proportional to the corresponding forces as defined in the text.

When masses are compared in accordance with the second principle two cases can arise. If the substance is of the same kind for all the bodies, the mass is experimentally found to be proportional to the *volume simply*. Now, in such a case, there can be no hesitation in taking the volume as a measure of the quantity of matter present (for each unit volume obviously contains the same amount of stuff); thus in this case the mass is proportional to the quantity of matter. When the materials are chemically different (*e.g.* let one be lead, the other iron), there is no obvious way of measuring the quantity of matter in a substance. It is agreed to consider it as being *always* measured by its mass. Equal volumes in general have different masses, and therefore, according to this definition, contain different quantities of matter.

It does not require any refined experiment to show that it is easier to set in motion a piece of wood than an equal volume of lead, and also easier to stop it. Hence one of the advantages that the use of lead for bullets has.

Density. The mass of unit volume of any substance is called its *density.* The density of water being taken as unity, that of aluminium is 2·6, and of lead is 11·35 (see tables at end).

If, as is possible, it should be proved that all substances are built up out of a single ultimate substance, it must be more tightly packed in some cases than in others.

Conservation of Matter. It is generally believed that the total mass in the universe is constant. This belief is referred to as the *Conservation of Matter.* It is based simply on the evidence that any changes we make—such as chemical combinations or separations, physical vaporisation, tempering, magnetisation, etc.—leave the mass unchanged.

Experimental Illustrations. The difficulty of separating the action of any two bodies from all others makes difficult a satisfactory elementary proof of these principles. All other actions must be made as small as possible, but there is always some surviving disturbance which introduces complication. Besides this, there is considerable difficulty in measuring an acceleration, especially when the force is very short-lived.

The following experiments and observations must be taken as only illustrating the principles; more accurate methods are reserved till a later stage.

To avoid the necessity of directly measuring accelerations, use may be made of the fact that the velocities gained in the same time by two interacting bodies must vary proportionately to the accelerations. Since the time of any interaction is the same for both bodies, it is sufficient to measure each velocity before and after the action; the change in velocity of each is proportional to the acceleration it has experienced. A convenient form of apparatus is known as Hicks' Ballistic Balance.

This consists of two carriers, each suspended by four threads as indicated in Fig. 9. Each carries an index which moves along a scale. The two are just in contact when at rest; but can be drawn aside to any extent and then let go.

Use is made of two facts:—First, that the time required to move from rest in any position to its central position is the same, however far these positions are apart. [This may at once be verified by drawing the two aside through *different* arcs and letting them go simultaneously—they will always strike in their lowest positions.] Secondly, as will be shown later, the velocity acquired in the movement is directly proportional to the arc of movement as long as the arc is small. (See p. 122.) Thus we have the means of measuring the velocity of each just prior to contact taking place between them; that is, just prior to their interaction. The arc through which each darts aside afterwards is also measured; this is similarly proportional to the velocity just after the interaction. Thus, the change in velocity of each can be calculated. Observation is

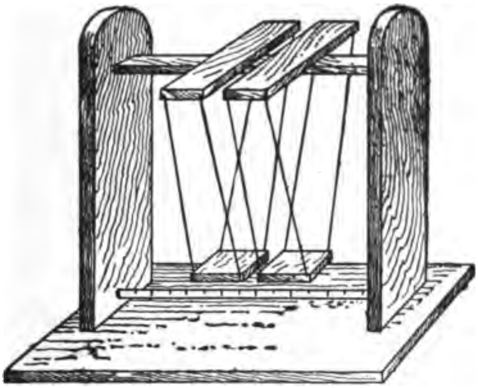


Fig. 9.

simplified if the carriers are made with spikes on the sides of contact, so that after impact they necessarily move as one. Various masses are placed in the two carriers, and observations made as above described in each case.

According to the definition given, the masses which collide in any trial (including, of course, that of the carrier) are inversely proportional to the changes in the velocity. The constancy of the mass can be verified by showing that it bears the same ratio to another given mass whatever is the actual change in velocity, and that when any three are compared together, two by two, consistent ratios are obtained. The data in any case are the following:—

	First Body.	Second Body.
Velocity just before impact	u_1	u_2
Velocity just after	v	v
Masses to be compared ...	M_1	M_2
Mass of carrier	m	

The velocities are to be measured positive when in one direction—say from left to right. If any of them are in the reverse direction they must be reckoned as negative.

Increase in velocity of second body $v - u_2$.
Increase in velocity of first body $v - u_1$.

Then

$$\frac{v - u_2}{v - u_1} = -\frac{M_1 + m}{M_2 + m} = -r \text{ (say).}$$

If the mass of the carrier is small compared with those placed in it, it may be neglected in preliminary experiments intended to illustrate the behaviour without complications. But finally it should be allowed for, by making an experiment with one of the carriers empty, the other containing a mass, M_1. Then

$$-\frac{M_1 + m}{m} = -r_0 \text{ (say)},$$

$$\text{or } m = \frac{M_1}{r_0 - 1},$$

where r_0 is the ratio of the changes in velocity in this case. Thus each of the bodies can be compared with any one of them taken as a standard; and the full verification of the law consists in showing that whatever the initial arcs of motion or with whatever other body each is experimentally compared, the masses bear fixed ratios to one another.

This is the most direct way of verifying the principle; unfortunately it is not susceptible of very great accuracy. However, if a student has the opportunity of experimenting with the apparatus, he will soon acquire a sound conception as to what the term mass really means. Experiments with it do not determine the force at any instant; to do this it would be necessary to measure the acceleration and not the total change of velocity. Since the duration of the impact is exceedingly short (of the order of one-thousandth of a second) it is not possible to do this.

Definition of Weight. We have seen that any body near the surface of the earth falls to the ground with nearly constant acceleration. This implies that the force upon it is, to the same degree, constant. This force is called the weight, or gravity, of the body; the fact of the attraction is generally called *gravitation.*

The value of the acceleration is not quite the same at all places, and varies with the height above sea-level. The extreme of the values that have been actually measured are 983·08 $\frac{\text{cms.}}{\text{sec.}^2}$ at Spitzbergen, and 978·03 at Para. In general, it increases as one passes from the equator to either pole: and probably varies inversely as the square of the distance from the centre, as soon as sufficient height is reached to remove the effect of local peculiarities of the earth.

Since the gravitational acceleration varies from place to place, the weight must do so also, and in the same proportion, unless the mass changes also. It is always taken for granted that the mass remains unchanged.

Forces can often be conveniently measured by causing them to *neutralise* the effect of another force. This assumes that the forces produce their effect independently of one another. The actual acceleration being zero, these balancing forces must produce equal but opposite accelerations, and must therefore themselves be equal and opposite. Thus, if a body is hung from a spiral spring (Fig. 10) it stretches it, and then comes to rest. Here a force called into play by the extension of the spring counteracts the weight of the body. If a scale be placed so that an index attached to the lower end of the spring can move along it, the extension of the spring can be measured. Provided that the properties of the spring remain permanent—which may be tested by showing that the *same body* continues to produce the same extension—such a spring may be used for comparing forces in the following way. First adjust several bodies to have the same weight by altering them till, when hung *singly* from the spring, the extensions are the same. Any two of these taken together have double the weight of each, and will produce a greater extension. Adjust another body till it produces the same greater extension; then this body is of double weight. In the same way a triple weight —that is, one equivalent to three of the initial ones, or to one initial and one double one together—can be obtained, and so on for any multiple. Such an arrangement is called a *spiral balance.*

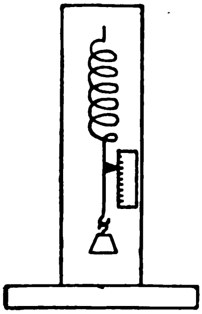


Fig. 10.

It will be found that the extensions are very nearly proportional to the weights producing them. It, however, would not do to *assume* this law, which expresses a fresh experimental fact. It is known as Hooke's Law.

Again, it is assumed that all other forces than gravitation are carefully excluded. If the stand be made of steel, and some of the bodies are steel, discordant results may be obtained, because forces arising from magnetisation may be superimposed.

Since force is measured by the product of mass into acceleration, we can write

$$F = ma,$$

where the quantities are represented by their initial letters. Let W be the weight of a body—*i.e.* the force with which the earth pulls it. The acceleration when it acts alone is that of gravity, which we denote by g; then

$$W = mg;$$

$$\text{therefore} \quad \frac{F}{W} = \frac{a}{g}.$$

This alternative equation merely states that forces are proportional to the accelerations they produce in a given mass. The drawback to comparing a force with a weight is that, as we have seen, the weight of a particular body varies as it moves from place to place; whereas the ratio of W to g, viz. m, is constant.

Units of Mass and Force. It is convenient to take some given mass and force as unitary standards, others being then expressed as multiples of them. The unit of acceleration we will consider as already fixed, although there are several to choose from; those of force and mass will not be independent of each other, on account of the equation, $F = ma$, connecting them. We may begin by defining the unit of mass; the unit of force will then be derived by means of this equation. The system of units so obtained is called an *absolute* system.

1. The international absolute system.

The centimetre and second are the units of space and time.

The unit of mass is one-thousandth that of a block of platinum-iridium, called the International Prototype Kilogramme, kept at the International Bureau at Sèvres. The unit itself is called one gram (or gramme).

The unit force is that which corresponds to an acceleration of one centimetre per second per second in a mass of one gram; it is called a *dyne*.

The gram was originally intended to be the mass of one cubic centimetre of water at its temperature of maximum density. Owing to difficulties of measuring a cubic centimetre of water and comparing it with the actual gram, it is not known to what extent the original intention is fulfilled, though the deviation is only very slight. No great advantage would accrue from such correspondence, and to-day the more satisfactory method is adopted of choosing a standard which can be expected to remain permanent if carefully preserved, rather than one which is perfectly constant theoretically, but which it is impossible to realise in practice.

If the unit force is chosen before that of mass the system is known as a *gravitational* system, because the unit chosen is always a particular weight.

2. The C. G. S. gravitational system.

The unit of force is the *weight* of the mass of one gram referred to above.

Unit mass is the mass in which this force produces an acceleration of one centimetre per second per second. It has received no name; it is obviously equal to 981 grams.

It is perhaps more common to take the weight of the kilogram as unit of force and the metre as that of length. This system is largely used by engineers on the Continent.

3. British gravitational system.

The weight of a block of platinum called the imperial pound is taken as the unit of force; we shall call it one pound weight. The unit of mass is that mass to which this unit force will give an acceleration of one foot per second per second. It has no name. Since a force of one pound weight so defined produces an acceleration of 32·2 feet/sec.^{2} in the block itself, it would produce unit acceleration in a *mass* 32·2 times that of the standard block. The *mass* of the block we shall speak of as *one pound.* This system is used by British engineers.

The relation between these units is that 453·6 grms. weight are contained in a pound weight.

4. A British absolute system has also been

devised, in which the unit mass is one pound; the unit force is then defined as being that which gives to a mass of one pound an acceleration of one foot per second per second, and is called one *poundal.* It has not come into use, however; for the international system is employed for all scientific work, and engineers find the gravitational system more convenient for them.

In making use of either of the equations,

$$F = ma,$$

$$\text{or} \quad \frac{F}{W} = \frac{a}{g},$$

it should be borne in mind that each is true, whichever of the above systems of units is consistently employed. In practice, however, the former is more convenient when absolute systems are used; the latter is so when use is made of a gravitational system, because it contains no reference to a mass, the unit of which has received no name. For greater clearness we write here in full the units to be employed in actual calculation.

Force in dynes = mass in grams × acceleration in cm. per sec. per sec.

Force in poundals = mass in lb. × acceleration in feet per sec. per sec.

$$\frac{\text{Force expressed in any unit thereof}}{\text{Weight expressed in same unit}} = \frac{\text{Acceleration expressed in any unit thereof}}{\text{Gravitational acceleration expressed in same unit}}$$

For example, if we want the force to be in dynes, the weight must also be expressed in dynes; if we want it to be in lb. wt., the weight must also be so expressed.

APPLICATIONS OF FUNDAMENTAL PRINCIPLES.

The illustrations given of the fundamental principles will now be further extended to various cases:

1. A block resting on a stationary horizontal table.

We shall assume that the earth pulls the block with the same force whether the table is interposed or not; in other words, that its weight is the same. That it is so is a noteworthy fact, which can be shown to be roughly true with a spiral balance (p. 43); more sensitive

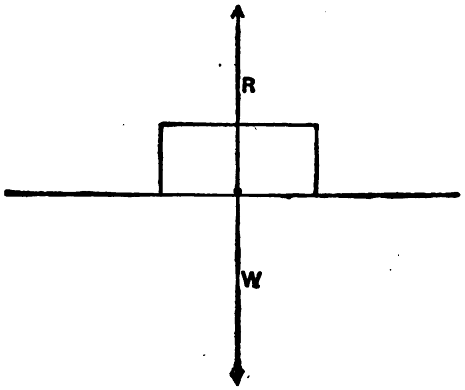


Fig. 11.

ways of comparing weights (see Balance, p. 265) have revealed no difference in weight. Thus one force acting on the block is W, its weight directed vertically downwards. But it is at rest; and it has no acceleration. An equal force to W, but opposite in direction, must be therefore present; this is provided by the table, which presses upwards on it, preventing gravity from producing motion. We shall denote it by R, and can write R = W.

The directions of the forces on the block are shown in Fig. 11. Since action and reaction are equal and opposite, a downward force, R, must act *on the table* due to the block. If the surface layers of the table consisted of springs, as shown in Fig. 12, all those under the block would be *compressed*. The action between the two bodies in all such cases is called a *pressure*.

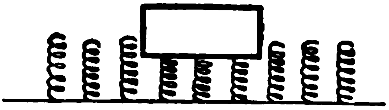


Fig. 12.

2. Block resting on a table which has an acceleration.

In this case one of the forces must be greater than the other. Let a be the acceleration measured positive if upwards. The upward force is R − W, and this equals mass × acceleration,

$$\text{or} \quad R - W = ma.$$
$$\text{But} \quad W = mg;$$
$$\therefore \quad R = m(g + a).$$

Thus the pressure is greater than before in the ratio

$$\frac{R}{W} = \frac{g + a}{g}.$$

In calculating from the formula use the same unit for a as for g (*any* unit so used will do); then R will be expressed in terms of the same unit as you use for W. If the acceleration is in reality *downwards* the same formula holds good, a negative numerical value being given to a.

Example: Let a be 2 feet/sec.^{2} upwards.

Now $g = 32$ feet/sec.^{2};

$$\therefore \quad \frac{R}{W} = \frac{34}{32}.$$

Hence if W = 16 lb. weight,

$$R = \frac{34 \times 16}{32} = 17 \text{ lb. weight.}$$

If W = 16 grms. weight,

R = 17 grms. weight.

If the velocity is constant, a is zero and R = W; it is only if there is acceleration that they differ.

3. A block held up by a string at rest (Fig. 13).

In this case gravity is counteracted by a force due to the string; we denote the force by T. Then, if there is no acceleration,

$$T = W.$$

If there is acceleration, a, upwards,

$$T - W = ma, \text{ and } W = mg;$$

$$\therefore \frac{T}{W} = \frac{g + a}{g}.$$

$$\text{and } T = \frac{W}{g}(g + a) = m(g + a).$$

The string is *stretched* owing to the block; if it were made of indiarubber or if it were a spiral spring the stretch would be easily visible. When a body is stretched by force it is said to be in *tension*; the action itself is also called a tension. The formula is the same as before, and requires no further comment.

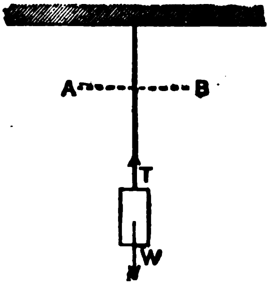


Fig. 13.

4. The tension is not the same at every part of the string, for, at any section, W is the whole weight sustained by it. Thus consider the two parts above and below the cross-section AB. The lower part has a weight W_1, say; this is counteracted by a tension T_1 arising from the upper part;

$$\text{thus, } \frac{T_1}{W_1} = \frac{g + a}{g} \text{ as before.}$$

Since W_1 includes the weight of part of the string as well as of the block, T_1 is greater than T for the same value of *a*.

5. Two masses fastened together by a hori-

zontal string and acted upon by external bodies (Fig. 14).

Let the external forces be horizontal and equal to F_1 and F_2 as shown, acting on the masses m_1 and m_2 respectively; gravity will also act vertically on each, but we will suppose its effect cancelled by the introduction of upward forces equal to the weights, so that we may consider horizontal motion alone.

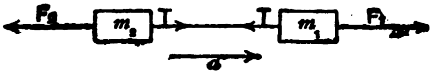


Fig. 14.

If the string remains tight and does not stretch, each body must move at any instant in the same way as the other—*i.e.* their displacements, velocities, and accelerations are the same. Let a be the common acceleration from left to right; F_1 and F_2 cannot be the only forces acting, for if they were, m_1 would have an acceleration of F_1/m_1 from left to right, and m_2 an acceleration of F_2/m_2 from right to left. *The string constrains the acceleration to be the same.* It pulls m_1 backward and m_2 forward. If the string be considered part of the body m_1 (say), the action may be taken as being between the bodies themselves. Let its value be T, forward on m_2, backward on m_1. Then the acceleration of m_1 is due to the difference between F_1 and T, or

$$F_1 - T = m_1 a.$$

Similarly, considering m_2,

$$T - F_2 = m_2 a.$$

Add these equations:

$$F_1 - F_2 = (m_1 + m_2)a.$$

Thus the two bodies move as a single body of mass equal to the sum of the masses acted upon by the external forces F_1 and F_2 alone. Forces (such as T) between different parts of a single system have no influence on the motion of the system; it is only the forces (such as F_1 and F_2) which arise from external bodies which are determinative of motion.

The value of the internal action T can be obtained by eliminating a from the two equations; the result is

$$T = \frac{F_1 m_2 + F_2 m_1}{m_1 + m_2}.$$

6. Bend the string over a peg so that the two parts are parallel to one another, and let the directions of the forces still be parallel to the string. The same equations as before will be obtained if we take a as being positive *downwards* on the right-hand side and *upwards* on the left (Fig. 15).

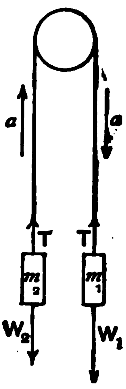


Fig. 15.

[The two tensions, T, shown in the figure, no longer represent action and corresponding reaction; indeed, since they are *in the same direction* that could not possibly be the case. The reactions to each of them are to be sought for at the peg, the effect of which is simply to reverse the *direction* of the string and the tension in it without altering the magnitude of the latter;—at least, this is so if the peg is perfectly smooth. The matter is more fully discussed later on.]

Further, let the forces F_1 and F_2 be the *weights* of the bodies; then

$$F_1 = W_1 = m_1g, \qquad F_2 = W_2 = m_2g,$$

and the equations become

$$m_1g - T = m_1a$$
$$T - m_2g = m_2a.$$
$$a = \frac{m_1g - m_2g}{m_1 + m_2} = \frac{m_1 - m_2}{m_1 + m_2}g.$$
$$T = \frac{m_1g \,.\, m_2 + m_2g \,.\, m_1}{m_1 + m_2} = \frac{2m_1m_2g}{m_1+m}$$
$$= \frac{2m_1}{m_1 + m_2}W_2 = \frac{2m_2}{m_1 + m_2}W_1.$$

In the equation for a the masses may be expressed in any unit, because they occur as a mere ratio, which must be the same in any system. The result will give a in terms of the same unit as that in terms of which g is expressed.

In the two last equations for T it is given as a ratio to either W_2 or W_1; it will be expressed in terms of the same unit as these are, any unit whatever being used for the masses in the expressions $m_1/(m_1 + m_2)$ and $m_2/(m_1 + m_2)$ respectively.

Or again, since the masses are proportional to the weights, we may write

$$T = \frac{2W_1W_2}{W_1 + W}$$

Example: Let the masses be $m_1 = 10$ grms., $m_2 = 6$ grms., then

$$a = \frac{10 - 6}{10 + 6}g = \frac{g}{4};$$

i.e. about 8 ft./sec.^{2}, or 245 cm./sec.^{2};

and $$T = \frac{2 \times 10}{10 + 6}W_2 = \frac{5}{4}W_2.$$

Now $$W_2 = 6 \text{ grms. weight};$$

$$\therefore T = \frac{30}{4} \text{ grms. weight.}$$

Or again, $$W_2 = 6 \times 981 \text{ dynes};$$

$$\therefore T = \frac{5 \times 6 \times 981}{4} \text{ dynes.}$$

In the above arrangement the acceleration becomes zero if $m_1 = m_2$; it becomes g if m_2 is zero; in all other cases it lies between these two values. Such an arrangement provides, therefore, a way of diluting gravity. Since the acceleration can be made small, measurements of it are more manageable than in the case of a freely falling body; and it would prove a most valuable means of experimenting on forces but for the complications which are necessarily present in any actual apparatus. These will be discussed in the chapters on Pulleys (p. 269) and Moments of Inertia (p. 209). Here we will give an outline of the way to eliminate the effects of the disturbances.

Firstly, by experiment it is found that the mass may be a little greater on one side than on the other, and yet no motion takes place;

this is due to the peg opposing the sliding of the string over it; if f is this difference in mass, fg is the corresponding force (Friction). The numerator in the expression for a, which represents the effective force, should be reduced by this amount—*i.e.* it should be

$$(m_1 - m_2 - f)g.$$

Secondly, in order to keep the friction small, the peg is replaced by a pulley mounted either on fixed bearings or on *friction wheels.* The wheel is set rotating when the string moves; this corresponds to an increase in the total mass moved, but contributes nothing to the force producing motion; that is, it affects the denominator alone, which becomes

$$m_1 + m_2 + \mu,$$

where μ is the effective mass (p. 210) of the pulley. Hence a more correct value for a is

$$a = \frac{(m_1 - m_2 - f)g}{m_1 + m_2 + \mu},$$

which may be contracted into

$$a = \frac{Rg}{M + \mu}.$$

In order to determine g, the acceleration a is obtained for a given value of R and M by measuring the distance which the right-hand weight descends in various times (as on p. 29). The value of M is then changed, R being kept the same; and the new acceleration a_1 is determined in the same way. Insert its value into the equation:

$$a_1 = \frac{Rg}{M_1 + \mu}.$$

Write these equations:

$$Rg = a(M + \mu),$$
$$Rg = a_1(M_1 + \mu),$$

and eliminate μ. The result is

$$g = \frac{aa_1(M - M_1)}{R(a_1 - a)}.$$

This apparatus is known as Atwood's machine; equal masses are placed in two pans, one on each side, and motion is produced by adding a small weight (the *rider*) to one side.

It might be thought that it would do equally well to keep the equal masses the same, and obtain a second acceleration for the elimination of μ by altering the rider. Although this is theoretically the case, yet the experiment is then badly arranged (in fact, hopelessly so), for the denominator in the expression for g then consists of a very small difference between two large terms, and it requires a very small error in the experimentally determined value of either of these terms to make a very large error in the result.

Change of Units.

1. To find the number of dynes in 1 lb. weight:

x dynes = 1 lb. weight = 453·6 grms. weight
= approx. 453·6 × 981 dynes = 445,000 dynes nearly

[*Note*. Since g is not the same in all parts of the earth, the relation required is variable.]

2. A certain force has the value 16 when grams, centimetres, and seconds are the fundamental units; what is its value when kilograms, metres, and minutes are employed?

Force is mass × acceleration.

$$16 \text{ grms.} \frac{\text{cms.}}{\text{sec.}^2} = x \frac{\text{kilograms, metres}}{\text{min.}^2}$$

$$= x \frac{1{,}000 \text{ grms.} \times 100 \text{ cm.}}{3{,}600 \text{ sec.}^2};$$

$$\therefore\ x = \frac{16 \times 3{,}600}{100{,}000} = \cdot 576.$$

EXAMPLES TO CHAPTER III.

1. A force produces an acceleration of 10 ft. per min.^{2} in a mass of 40 lb. What acceleration will it produce in 1 cwt.?

2. A force gives a velocity of 6 ft. per min. to a mass of 20 lb. in four seconds. Compare it with a pound weight when $g = 32 \cdot 2$.

3. Find the number of dynes in a pound weight when $g = 32 \cdot 2$.

4. A body of 4 grams mass moves 50 cms. from rest in ten seconds under the action of a constant force. What is the value of the force?

5. A cricketer stops a ball (mass = $3\frac{1}{2}$ ozs.) which is moving horizontally with a velocity of 400 cms. per sec., (1) in 2 secs., (2) in 50 cms. What force does he exert in each case on the average?

6. If the ball were falling vertically, what force must he exert?

7. A man in a lift can just hold up a weight of 56 lb. when the lift is at rest. How much can he raise when the lift has (1) a constant velocity upwards of 2 ft. per sec.; (2) an acceleration upwards of 3 ft. per sec. per sec.; (3) the same acceleration downwards; (4) a downward acceleration of g.

Questions on Ideal Atwood's Machine.

8. The masses in an Atwood's machine are 5 and 6 grams. Find their accelerations and the tension in the string.

9. Show that the tension in the string is always greater than one weight and less than the other.

10. If the weights are contained in scale-pans (whose weight can be neglected), find the pressures between the weights and pans when in free motion.

11. If, when the masses are 20 and 21, the acceleration is found to be one-half foot per sec. per sec., find the effective mass of the pulley wheel.

12. If, in question 8, the string were to break when the heavier mass had descended 3 ft., find the velocity of each mass two seconds later.

13. If the rising weight in 11 were to pick up another on its way, what influence would this have on the motion?

14. If there are four moving weights, one attached on each side below the ordinary weight by a string between them, find the tension in each part of the string when motion is taking place.

CHAPTER IV

MOMENTUM

THE equality of action and reaction is capable of being expressed in another way. We have already replaced the accelerations by the changes in velocity in equal times. Let the velocities of two masses, m_1 and m_2, before the action be u_1 and u_2, and after a given time be v_1 and v_2; then, if the changes are produced by their mutual action,

$$m_1(v_1 - u_1) = - m_2(v_2 - u_2)$$

or

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$

The product mu is called the *momentum* of the body whose mass is m and velocity u. Hence the above equation states that the sum of the momenta of two bodies remains unchanged by their interaction. This fact is referred to as the Principle of the Conservation of Momentum.

In interpreting the Principle, the momentum is to be taken as having the same sign and direction as the velocity to which it corresponds.

Since acceleration is the rate of increase of velocity, force must be equal to the rate of increase of momentum; that is,

$$F = ma = \frac{m(v-v_0)}{t} = \frac{mv-mv_0}{t}.$$

The total change in the momentum of a body is called an *Impulse* or *Blow.* If the force throughout the action is constant, the Impulse $= mv - mv_0 = Ft$. If the force is not constant throughout, the whole time may be divided into intervals so short that in each of them it sensibly is so; the product of force into the short interval is the impulse during the interval. Since the total change of momentum is equal to the sum of the successive changes, the total impulse is equal to the sum of the successive impulses.

When collision takes place between two bodies, such as billiard balls or the carriers in Hicks' Balance (p. 38), the principle of the Conservation of Momentum is usually not sufficient of itself to determine the dependence upon one another of the velocities before and after the collision. The masses of the colliding bodies and the velocities just before collision being supposed known, there are the two subsequent velocities to be determined; and, consequently, two equations are necessary for ascertaining them. The motions of the several parts of the colliding bodies are too complicated to be dealt with individually; to avoid the necessity of attempting this, recourse is had to experiment, with the object of finding an empirical law connecting the velocities. The first thoroughgoing experiments were made by Newton, on balls of different materials, hung as pendulums and colliding together. Newton found that for the same two bodies the relative velocity after collision bears a constant negative ratio to the relative velocity before. Thus, if u_1 and u_2 are the velocities before collision, and v_1 and v_2 the values subsequent to collision,

$$\frac{v_1-v_2}{u_1-u_2} = \text{negative constant} = -e \text{ (say).}$$

The positive constant, e, is now called the *coefficient of restitution* or *of rebound.* Its value depends upon the material of the bodies and also upon their shape.

More recent experiments (Hodgkinson) have shown that e is not rigorously constant, but decreases with increase in the relative velocity.

Taking it as a constant for present purposes, a knowledge of the velocities after collision is obtained by combining the last equation with that given by the principle of conservation. Thus, if m_1 and m_2 are the masses of the two bodies,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2u_2$$

and

$$-eu_1 + eu_2 = v_1 - v_2.$$

By eliminating v_2 we get

$$v_1 = \frac{m_1u_1 + m_2u_2 + em_2(u_2 - u_1)}{m_1 + m_2}.$$

By eliminating v_1:

$$v_2 = \frac{m_1u_1 + m_2u_2 - em_1(u_2 - u_1)}{m_1 + m_2}.$$

When $e = o$, $v_1 = v_2$—that is, the two bodies move with a common velocity after the collision—they are then said to be *inelastic.* Two masses of damp clay form a nearly perfect example. The constant e always lies between zero and unity; in the latter case the bodies would be *perfectly elastic*, but none fulfil this condition in perfection. The values for various substances are given below, on the authority of Hodgkinson (*Report to the British Association*, 1834):

Cast Iron Balls	·66
Elm Balls	·60
Lead	·20
Glass	·94
Ivory	·81

A further examination of the phenomenon shows that for all partially elastic bodies it is divisible into two parts; in the first part the interaction is such as to bring the relative velocity to zero. In the case of a perfectly inelastic body the action is then at an end. Both are then moving with a velocity

$$(m_1u_1 + m_2u_2)/(m_1 + m_2).$$

In all other cases the tendency of the bodies to recover their original form causes the relative velocity to once more increase, but with opposite sign. If the recovery were perfect the original relative velocity would be restored, but with change in sign; this would correspond to $e = 1$. In any actual case some permanent deformation takes place; in other words, the recovery is incomplete—that is, $e < 1$. In the same way we may divide the impulse into two parts. In the first it is equal to

$$m_1\left(\frac{m_1u_1 + m_2u_2}{m_1 + m_2} - u_1\right) \text{ or } \frac{m_1m_2}{m_1 + m_2}(u_2 - u_1);$$

in the second part it is

$$m_1\left(v_1 - \frac{m_1u_1 + m_2u_2}{m_1 + m_2}\right); \text{ that is, } e\frac{m_1m_2}{m_1 + m_2}(u_2 - u_1);$$

that is, the impulse during the second stage is e times that during the first.

The character of the changes that go on during the collision can to some extent be studied on hollow indiarubber balls. The change of form which takes place when they are pressed together and the recoil on releasing them are of the same kind as when two balls of glass or steel collide; but in the case of such substances as steel the change of shape is not nearly so great.

As an illustration of perfectly inelastic substances, there is nothing better than damp clay. Two equal lumps of it placed in carriers on a Hicks' Balance, and projecting over the edges of them, come absolutely to rest when initially drawn aside through equal arcs. The change of shape produced is quite conspicuous.

Repeated impact between a hammer and nails splays the surface disadvantageously; impact between a hammer and a rivet splays it usefully.

When a body falls to the ground it rebounds to a greater or less extent; it must not be assumed, however, that the value of e has the same value as for the collison of two spheres. A lead ball falling on a table scarcely rebounds at all, while two lead balls hung as pendulums and allowed to collide have a value of ·2 for e. For any given value of e the height of rebound can be calculated. For if the relative velocity is v_0 (measured downwards) just before, and v_1 (measured upwards) just after, $v_1 = ev_0$. These relative velocities are the velocities observed by *an observer on the earth*; hence if h_0 is the initial height and h_1 the height to which the ball rises after the first rebound $v_1^2/v_0^2 = h_1/h_0$; hence $h_1 = h_0e^2$.

If it be again allowed to fall and rebound repeatedly, and if h_2, h_3 . . . are the succeeding heights, the whole *length of path* traversed is:

$$\begin{aligned} & h_0 + 2h_1 + 2h_2 + 2h_3 + \dots \textit{ ad inf.} \\ = {} & h_0(1 + 2e^2 + 2e^4 + 2e^6 + \dots \textit{ ad inf.}) \\ = {} & h_0\left(\frac{2}{1 - e^2} - 1\right) = h_0\,\frac{1 + e^2}{1 - e^2} \end{aligned}$$

Since the time of falling on each occasion is equal to the square root of $(2h/g)$ when h is the value of the particular height, the total time must be:

$$\frac{2h_0}{g}(1 + 2e + 2e^2 + 2e^3 + \ldots)$$
$$= \frac{2h_0}{g}\left(\frac{2}{1-e} - 1\right) = \frac{2h_0}{g} \cdot \frac{1+e}{1-e}$$

The student should examine this case carefully, as it is a good example of a limiting value. Although the number of rebounds is infinite, the total time occupied is finite; for the duration of succeeding up and down journeys gets smaller according to such a law that the sum of their durations forms a series which is converging. For example, if $e = \frac{1}{2}$, the total time is only three times that taken in the first fall.

It should also be observed that in this case of impact where only the *relative* velocities were wanted, it is unnecessary to introduce the principle of momentum. If, however, the velocities are required of ball and *earth* with reference to some point whose motion is unaltered during their movement, the principle must be introduced as usual. The earth gains precisely the same momentum as that which the ball loses; the velocity gained by it is very small, however, because the mass is huge. The relative velocity v of the ball is, therefore, very nearly indeed the velocity with regard to the undisturbed point. The velocity gained by the earth bears to this the ratio of the mass of the ball to its own mass.

Pressure due to falling chain. If a chain be lowered so as to coil up on the ground, each part has motion just before it reaches the ground, but its velocity is then taken away from it by impact. The corresponding decrease of momentum causes the force on the ground to be greater than the weight of that portion which has at any time come to rest; the excess must equal the rate of decrease of momentum of the chain. For example, suppose the chain travels at the rate v, and μ is its mass per unit length; the rate of decrease of momentum is $\mu v . v$, for a length v will be reduced to rest in each unit of time. If m is the mass that at any time is on the table, the total force due to the table is $mg + \mu v^2$.

The only essential difference between this and the other cases of impact which have been discussed is that in those the action was nearly instantaneous, while in the above case there is a *continuous* change of momentum. As another instance we may mention that when a load of bricks is "tipped" the force on the ground is at any instant greater than the weight of the bricks that are on it.

EXAMPLES TO CHAPTER IV.

1. Two balls moving with velocities 10 ft. and 4 ft. per sec. respectively, in the same direction, collide, and their velocities become 9 and 5. Compare their masses.

$$\left.\begin{array}{l}\text{Increase of velocity of first} = -1\\ \text{Increase of velocity of second} = 1\end{array}\right\}\text{Ratio of masses} = -\frac{1}{-1} = 1$$

2. If the initial velocity had been 10 and -4, and the final -9 and 5, what would be the ratio of the masses?

3. An engine moving with a velocity of 8 miles per hour collides with a stationary truck, and the two then move with a velocity of 7½ miles per hour. Compare the masses of engine and truck.

4. A shot strikes and becomes embedded in a suspended target. The masses of the shot and target are as 1 to 100, and the target moves with a velocity of 10 ft. per sec. as soon as the shot is imbedded. What was the velocity of the shot on striking?

5. Masses of 3 and 6 are moving with velocities 14 and 8 just before collision. Is it possible for their velocities just after collision to be 8 and 14 (in either direction)?

6. Two masses of 4 and 3 move toward one another with velocities 10 and 5. If they collide and then stick together, what is their velocity after collision?

7. If the co-efficient of restitution in question 6 were two-thirds, what would the final velocities be?

8. A ball bounces repeatedly from a table after falling from a height of 8 feet. How long will the rebounding last, and what is the total length of path traversed, the value of e being one-half?

9. A ball of 5 lb., moving with a velocity 6, impinges directly upon a ball of 10 lb. moving in the same direction with a velocity 4; the velocity of the latter after impact is 5. What is the value of e?

10. If the momenta of two bodies are equal and opposite before impact, show that they are equal afterwards.

CHAPTER V

ADDITION OF VECTORS

ANY quantity for which both magnitude and direction require specification is called a *vector*; displacement, velocity, and acceleration are vectors.

We have so far only considered displacements which take place along a single line. The same definition which was given (p. 10) holds in every case. Hence if a body moves from A to B, then to C and D in succession, the displacements in the various intervals are the straight lines $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ respectively. In the whole interval the displacement is $\overline{AD}$; this is called the resultant displacement, and the rest are components. The rule connecting them evidently is: Draw the component displacements end to end so as to form the sides of an unclosed polygon; the line joining the initial and final points on these lines so as to close the polygon is the resultant.

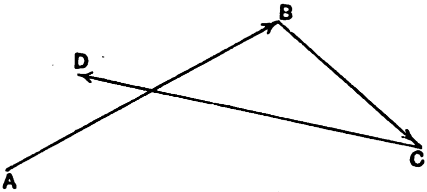


Fig. 16.

Such a figure is called a polygon of displacements; if there are only three it becomes a triangle. The process indicated is a particular kind of addition, differing from algebraic addition in so far as attention is paid to *directions* as well as magnitudes. It is called vector addition, and is denoted by the symbol $\mathbin{+\!\!\!\!+}$; thus the above statement is written

$$\mathrm{AD} = \mathrm{AB} \mathbin{+\!\!\!\!+} \mathrm{BC} \mathbin{+\!\!\!\!+} \mathrm{CD}.$$

We have supposed the components to occur successively, but they can occur simultaneously. Thus, a body may move from A to B on a card which is itself translated through a displacement $\overline{\mathrm{BC}}$; the body will reach C as the result of the two motions, and the resultant displacement is $\overline{\mathrm{AC}}$.

Similarly the actual or resultant velocity is connected with the velocities of the component motions by the same rule; for $\overline{\mathrm{AB}}$ and $\overline{\mathrm{BC}}$ may be chosen so as to represent the displacements in unit time if the velocities are constant, and $\overline{\mathrm{AC}}$ will then be the resultant displacement per unit time—*i.e.* the resultant velocity.

If the velocities are not constant, let the scale of the diagram be magnified indefinitely, so that AB, BC, and AC are displacements in the *same* limitingly short time. Then these are proportional to the respective velocities, and the rule still holds good.

The resultant can obviously be obtained from the components by making a diagram to scale and measuring it. Since careful drawing is required to obtain even rough accuracy, it is better to calculate the resultant by the rules connecting the sides and angles of a triangle when there are two components, or, in general, of a polygon. A short summary of the rules required is given below; these must be supplemented from Treatises on Trigonometry where the rules are proved in full.

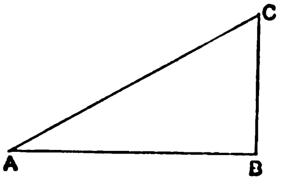


Fig. 17.

Let ABC be a right-angled triangle with the right angle at B. The ratio $\frac{\overline{CB}}{\overline{AC}}$ is called the sine of the angle A, and is written sin A.

The ratio $\frac{\overline{AB}}{\overline{AC}}$ is called the cosine of the angle A, and is written cos A.

The ratio $\frac{\overline{BC}}{\overline{AB}}$ is called the tangent of the angle A, and is written tan A.

Since $\frac{\overline{BC}}{\overline{AB}} = \frac{\overline{BC}/\overline{AC}}{\overline{AB}/\overline{AC}}$, it follows that

$$\tan A = \frac{\sin A}{\cos A}.$$

Since $(\overline{BC})^2 + (\overline{AB})^2 = (\overline{AC})^2$

$$\left(\frac{\overline{BC}}{\overline{AC}}\right)^2 + \left(\frac{\overline{AB}}{\overline{AC}}\right)^2 = 1$$

or $\sin^2 A + \cos^2 A = 1$

(Here $\sin^2 A$ stands for the square of $\sin A$, and $\cos^2 A$ for the square of $\cos A$; similarly the square of $\tan A$ is written $\tan^2 A$.)

Similarly $$\frac{\overline{AB}}{\overline{AC}} = \sin C; \quad \frac{\overline{BC}}{\overline{AC}} = \cos C;$$

$$\frac{\overline{AB}}{\overline{BC}} = \tan C.$$

But these ratios are respectively the cosine, sine, and the ***reciprocal of*** the tangent of A; further, A and C add up to a right angle.

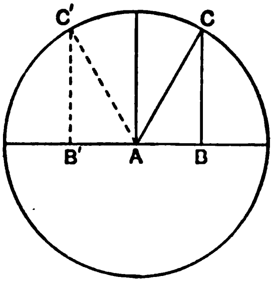


Fig. 18.

Thus, if two angles add up to a right angle, the sine of one equals the cosine of the other, etc.

These definitions are all based on the properties of a right-angled triangle. Their utility arises in expressing the properties of *any* triangle in terms of them. Since, however, no right-angled triangle contains an angle greater than a right angle, it is first necessary to define what meaning shall be given to the sine, cosine, or tangent of such an angle.

Consider a circle of radius unity (Fig. 18). Since $\overline{AC}$ is unity, the sine of A in the right-angled triangle ABC is $\overline{BC}$, and the cosine is $\overline{AB}$. Let the radius AC sweep round in a counter-clockwise sense, so as to increase the angle A. When this angle equals 90°, BC = 1 and AB = 0. If the angle A increase still further, so that C is in the position C^1, the sine of C^1AB is taken as being equal to $\overline{C^1B^1}$, and is therefore equal to the sine of the angle C^1AB^1; the cosine is taken equal to $\overline{AB^1}$, and is numerically equal to the cosine of the angle C^1AB^1; but since it is drawn in the opposite sense to $\overline{AB}$, it is considered to be negative. Hence, in words,

If two angles add up to 180° their sines are equal to one another, while their cosines are equal but of opposite sign.

$$\begin{aligned} \textit{i.e.}\quad \sin\theta &= \sin(180-\theta) \\ \cos\theta &= -\cos(180-\theta). \end{aligned}$$

Since the tangent of an angle is the ratio of its sine to its cosine, $\tan\theta = -\tan(180-\theta)$.

The following are a few values of the trigonometrical ratios:—

Case i.

$$\begin{aligned} A &= 45^\circ. \\ \text{Then } C &= 45^\circ \\ \text{and } AB &= BC. \\ \therefore\ AC &= AB\sqrt{2} \\ &= BC\sqrt{2}. \\ \therefore\ \sin 45^\circ &= \frac{1}{\sqrt{2}}, \\ \cos 45^\circ &= \frac{1}{\sqrt{2}}, \\ \tan 45^\circ &= 1. \end{aligned}$$

Case ii. $A = 60°$ (Fig. 19).

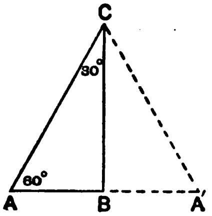


Fig. 19.

Turn the triangle ABC over the line BC so as to obtain an equal triangle CA^1B on the opposite side; then the angle $ACB =$ the angle $A^1CB = 30°$, and $AB = BA^1$, $\therefore$ the triangle ACA^1 is equilateral and $\overline{AB} = \frac{1}{2}\,\overline{AA^1} = \frac{1}{2}\,\overline{AC}$.

$$\therefore \cos 60° = \frac{AB}{AC} = \frac{1}{2} = \sin 30°;$$

$$\sin 30° = \frac{BC}{AC} = \sqrt{\frac{4-1}{4}} = \cos 60° = \frac{\sqrt{3}}{2}.$$

Case iii. $\sin 90° = \cos(\text{zero}) = 1$; and $\sin(\text{zero}) = \cos 90° = \text{zero}$.

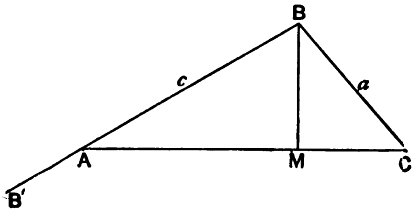


Fig. 20.

We will now apply these ratios to any triangle. Let ABC be any triangle. Drop a perpendicular from B to $\overline{AC}$, meeting it in M; then BMA and BMC are right-angled triangles.

$$\text{Hence } \frac{\overline{BM}}{\overline{AB}} = \sin A$$

$$\frac{\overline{BM}}{\overline{BC}} = \sin C.$$

$$\therefore \frac{\sin A}{\sin C} = \frac{\overline{BC}}{\overline{AB}} = \frac{a}{c}.$$

By dropping a perpendicular from A on $\overline{BC}$, produced if necessary, it can be similarly shown that

$$\frac{\sin C}{\sin B} = \frac{\overline{AB}}{\overline{AC}} = \frac{c}{b};$$

that is, the three sides of any triangle are proportional respectively to the sines of the angles opposite to them.

Again,

$$\begin{aligned}(BC)^2 &= (\overline{BM})^2 + (\overline{MC})^2\\ &= (\overline{AB})^2 - (\overline{AM})^2 + (\overline{AC} - \overline{AM})^2\\ &= (\overline{AB})^2 + (\overline{AC})^2 - 2(\overline{AC})(\overline{AM})\\ &= (\overline{AB})^2 + (\overline{AC})^2 - 2(\overline{AC})(\overline{AB}) \cos A,\end{aligned}$$

or representing the sides by small letters corresponding to the angles opposite to them

$$a^2 = c^2 + b^2 - 2cb \cos A.$$

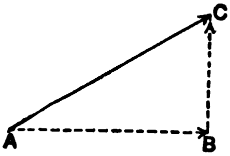
A is the internal angle BAC; the same result expressed in terms of the external angle B^1AC is

$$a^2 = c^2 + b^2 + 2cb \cos B^1AC.$$

We have specified in degrees the angles mentioned above. In theoretical investigations this is not the most convenient unit; it is preferable to take as unit angle that which is subtended at the centre of a circle by an arc equal in length to its radius. This angle is called a *radian.* Since the circumference of a circle has a length $2\pi r$ where r is the radius, and π is the never-ending expression 3·14159265 . . . it follows that 360° must equal 2π radians, and consequently

$$1 \text{ radian} = \frac{360}{2\pi} \text{ degrees} = 57{\cdot}3 \text{ degrees nearly.}$$

Resolution of Vectors.—The operation of finding components which are together equivalent to any vector is known as *resolution.*



Fg. 21.

Any vector can be resolved into two components in an indefinite number of ways; for it can form the third side of an infinite number of triangles. In the cases in which the two components are at right angles to one another they are called *rectangular* components. Even of these there are an infinite number of pairs; for it is clear that a perpendicular $\overline{CB}$ can be dropped on $\overline{AB}$ whatever the direction of the latter may be. The equations which give the rectangular components in any given direction are:

$$\overline{BC} = \overline{AC} . \sin CAB,$$
$$\overline{AB} = \overline{AC} . \cos CAB.$$

Considerable use is made of this mode of resolution of vectors.

Composition of more than Two Vectors. When there are more than two to be compounded the figure becomes a polygon, and the following method of composition is easiest:

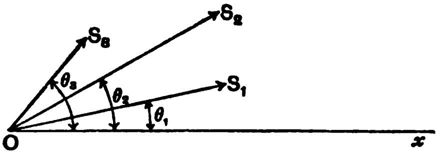


Fig. 22.

Let S_1, S_2, S_3, be three vectors making angles θ_1, θ_2, θ_3, with a given line of reference, Ox. Resolve each of these into two components, one along Ox and the other at right angles to it. The components along Ox are $S_1 \cos \theta_1$, $S_2 \cos \theta_2$, $S_3 \cos \theta_3$: these three may be added algebraically because they are all codirectional; let X be their algebraic sum. The components at right angles to these are $S_1 \sin \theta_1$, $S_2 \sin \theta_2$, $S_3 \sin \theta_3$; and being codirectional may be added together algebraically; let Y be their sum. We have thus reduced the vectors to two at right angles to one another, and their resultant R is equal to $\sqrt{X^2 + Y^2}$.

This method is obviously applicable to any number of vectors, where X is put equal to $\Sigma(S \cos \theta)$ and Y is put equal to

$$\Sigma(S \sin \theta).$$

To find the direction of the resultant we have $\tan \phi = \frac{Y}{X}$, where ϕ is the angle the resultant makes with X (Fig. 23).

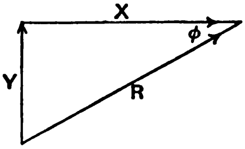


Fig. 23.

Note: Some difficulty will be found at first in deciding on the direction when either Y or X or both are negative. It is best for the beginner to first calculate X and Y; then to represent them in a drawing, drawn in correct sense; no difficulty will be felt in then deciding on the direction of the resultant. Thus, if X is negative, Y positive, the figure would be as shown (Fig. 24).

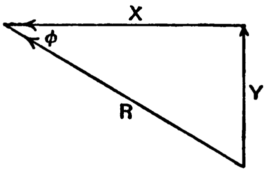


Fig. 24.

The numerical value of the ratio of Y to X will be the tangent of the angle ϕ in this figure.

Examples: 1. Find the resultant of two vectors, 9 and 3, making an angle of 150° with one another. (The student should draw the appropriate figure.)

In this case

$$R^2 = 9^2 + 3^2 - 2 \times 3 \times 9 \times \cos 30^\circ$$
$$= 81 + 9 - 54 \times \frac{\sqrt{3}}{2}$$
$$R = 6{\cdot}57 \text{ nearly.}$$

Further, to find the direction of R we have

$$\frac{\sin\phi}{\sin 30^\circ} = \frac{3}{\text{R}} = \frac{3}{6{\cdot}57}$$

$$\sin\phi = \frac{3}{13{\cdot}14} = {\cdot}228$$

By consulting a book of tables ϕ is found to be about 13° 11′.

Or thus: Take the direction of the vector 9 as the direction of x:

$$\begin{aligned} \text{X} &= 9\cos 0^\circ + 3\cos 150^\circ \\ &= 9 - 3\cos 30^\circ = 9 - \frac{3\sqrt{3}}{2}, \\ \text{Y} &= 9\sin 0^\circ + 3\sin 150^\circ \\ &= 0 + 3\sin 30^\circ = \frac{3}{2}, \end{aligned}$$

$$\begin{aligned} \text{R}^2 = \text{X}^2 + \text{Y}^2 &= 81 + \frac{9\times 3}{4} - 27\sqrt{3} + \frac{9}{4} \\ &= 90 - 27\sqrt{3} \quad \text{as before.} \end{aligned}$$

Again, $\tan\phi = \dfrac{\text{Y}}{\text{X}} = \dfrac{3}{18 - 3\sqrt{3}} = {\cdot}2343$, whence, by consulting tables, ϕ is found to be 13° 11′ as before.

If we take the direction of the vector 3 as direction of x,

$$\begin{aligned} \text{X}_1 &= 3\cos 0^\circ + 9\cos 150^\circ \\ &= 3 - 9\cos 30^\circ = 3 - \frac{9\sqrt{3}}{2}, \\ \text{Y}_1 &= 3\sin 0^\circ + 9\sin 150^\circ \\ &= 0 + 9\sin 30^\circ = 4\tfrac{1}{2}, \end{aligned}$$

which give $\text{X}_1^2 + \text{Y}_1^2$ the value previously obtained.

In this case X_1 is negative and Y_1 is positive; indicating that the angle ψ which the resultant makes with the direction of 9 is between 90° and 180°. Hence the triangle formed by X, Y, and R resembles Fig. 24.

2. Find the resultant of 12, 4, 6, 3, making angles of 30°, 45°, 60°, and 90°, with a given line Ox.

$$X = 12 \cos 30 + 4 \cos 45^\circ + 6 \cos 60^\circ + 3 \cos 90^\circ$$
$$= 6\sqrt{3} + 2\sqrt{2} + 3 + 0.$$
$$= 16{\cdot}22.$$
$$Y = 12 \sin 30^\circ + 4 \sin 45^\circ + 6 \sin 60^\circ + 3 \sin 90^\circ.$$
$$= 6 + 2\sqrt{2} + 3\sqrt{3} + 3$$
$$= 17{\cdot}02.$$

Whence $R = \sqrt{(16{\cdot}22)^2 + (17{\cdot}02)^2}$
$$= 23{\cdot}51$$

and $\tan\phi = \dfrac{17{\cdot}02}{16{\cdot}22} = 1{\cdot}05$
$$\therefore \phi = 46^\circ\ 30'.$$

Composition of Forces. Since a force is a constant into the acceleration it produces in a given mass, a single force that will produce the same acceleration as two given forces is obtained by the same rule as the resultant acceleration itself; for the figure representing the forces is similar to the figure representing the corresponding accelerations, but is m times its size where m is the mass of the body. Or in symbols

$$\begin{aligned} \text{since}\quad A &= a_1 +\!\!\!+\, a_2 \\ \therefore\quad mA &= ma_1 +\!\!\!+\, ma_2 \text{ (by similar figures)} \\ \text{or}\quad F &= F_1 +\!\!\!+\, F_2 \end{aligned}$$

where F is a single force which, acting alone, produces the same acceleration as F_1 and F_2 acting together. This single force is called the *resultant* of the two components.

Since this theorem is true for two forces, it must also be true for any number.

EXAMPLES TO CHAPTER V.

Note: Both magnitude and direction of a vector must be determined.

1. Find the resultant of the following vectors:

(1) 4 and 6 at 30° with one another
(2) 4 and 6 at 150° " " "
(3) 5 and 3 at 45° " " "

2. Find the resultant of 44 feet per second toward the north west, and 60 miles per hour due east.

3. Show that the resultant of two vectors cannot be greater than their arithmetic sum or less than their arithmetic difference.

4. Show that the resultant of two equal vectors can be numerically equal to each of them.

5. What must be the angle between two vectors, one of which is twice the other, in order that the resultant may be numerically equal to the larger?

6. A body travels six miles toward the north-west, then six miles due south, and finally four miles due east; what is the total displacement?

7. Find three pairs of possible rectangular components of a displacement of 12 feet.

8. Two trains, A and B, are travelling at an angle of 45° with one another and with velocities of 40 and 60 miles per hour; if the former is gaining speed at the rate of four miles per hour per hour, and the second is losing it at the same rate, find the relative velocity and also the relative acceleration of A to B.

9. Explain why the splashes of rain on the windows of a moving carriage are inclined to the vertical when the rain itself is falling vertically, and show how to calculate the relation between the velocities of the raindrops and carriage from a measurement of the angle of inclination.

10. A force of 100 dynes acts vertically on a 10 gram mass, and another force of 40 dynes acts horizontally upon it; what is the acceleration when both forces act together?

11. Two boats, A initially one mile due west of B, are sailing, A eastwards with a velocity of four miles per hour, and B southwards with a velocity of six miles per hour. Find their shortest hailing distance, the positions of each when it is shortest, and the time of reaching them.

CHAPTER VI

EQUILIBRIUM OF A PARTICLE

SINCE the theorems which were proved in Chapter III. hold good for a body of any size, we are warranted in applying them to a body so small that it may be considered as confined to a single point. Such a limitingly small body is called a *particle.* The advantage arising from the introduction of this conception is that a particle is incapable of rotation, or perhaps it is more accurate to state that its rotation has a vanishingly small dynamical significance. To restrict ourselves to dealing with a particle is equivalent therefore to the consideration only of the effect of forces all of which act at one point. We are freed thereby from the consideration of the complications which may enter into the motion of a body, and which depend upon the particular positions at which the forces act. The conception is useful, moreover, inasmuch as a body of finite size may be considered as built up of particles, subject not only to the action of external bodies but to mutual action amongst themselves; and we shall see that the most complicated motion of a body can be deduced (theoretically at any rate) from the simple *translatory* motion of its constituent particles.

Statics of a Particle.—If a particle has no acceleration—whether it is moving or not—the resultant of the forces acting on it must be zero. It is then said to be in equilibrium. It follows that the polygon of forces must then be a closed one. Thus if there are three forces -the lines representing them must, when drawn end to end in succession, form a closed triangle. If a polygon be drawn with its sides parallel to the forces, it neither follows that they will also be proportional to them nor conversely. In the case of a triangle, however, if its sides be parallel to the three forces keeping a particle in equilibrium they will also be proportional to them; again, if they are proportional to them they must make the same angle with one another as the given forces do, but are not necessarily parallel to them; in the last case parallelism can be brought about by rotating the triangle. These theorems are left to be deduced by the student, who should draw figures representing various possible cases.

Again, since the resultant force = $\sqrt{X^2 + Y^2}$ where X and Y (see p. 79) are respectively the sums of the components of each of the forces in each of two directions at right angles, X and Y must be *separately* zero. For X^2 and Y^2 are necessarily positive whether X or Y is positive or negative; thus it is impossible for $X^2 + Y^2$ to be zero unless X and Y are each zero.

The restriction that the directions of resolution shall be at right angles is unnecessary though it is usually most convenient to choose them so; it is sufficient that their directions shall not be the same.

In order to become convinced of the truth of this statement the student need only try and construct a closed figure from two lines whose directions are not the same.

The theorem may be at once applied to a body of finite size, it being tacitly understood that the forces are so applied that only translation can take place; it has also a more general application as will be shown at a later stage (p. 167).

Although, if a particle is *permanently at rest* the resultant force must be zero, the converse is not necessarily true. In fact, no theorem in respect to forces alone can indicate whether a body is at rest or not. Forces are not concerned with motion but with its mode of change.

Some examples will now be given:—

1. A particle of weight, W, hung up by a string, is pulled aside by a horizontal string; find the relation between the positions of and forces applied by means of the strings. The forces are as shown in the diagram (Fig. 25).

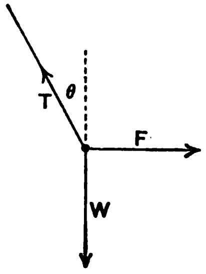


Fig. 25.

Resolve them vertically and equate the sum of their components to zero.

$$T\cos\theta - W = 0.$$

Resolve horizontally and equate sum of components to zero.

$$T\sin\theta - F = 0.$$

$$\text{Thus } T = \frac{W}{\cos\theta} \text{ and } F = W\tan\theta.$$

Hence, if θ and W are given, T and F can be calculated.

2. A particle of weight, W, is kept at rest on a *smooth* inclined plane by a force, P, parallel to the plane; find the force R between the plane and the particle, and the force P.

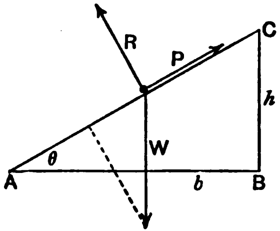


Fig. 26.

By a *smooth* plane is meant one which can only exert a force perpendicularly to its surface upon a body in contact with it.

The forces are as shown in Fig. 26. Besides the weight, W, and the applied force, P, there is a pressure, R, between the plane and the particle. It is this force which prevents the body from moving downwards through the plane.

Resolve parallel to the plane and equate sum of components to zero.

$$P - W\sin\theta = 0.$$

Resolve perpendicularly to plane.

$$R - W\cos\theta = 0.$$

Let h be the height of the plane, l the length $\overline{AC}$, and b the *base* or horizontal length; then $\sin\theta = \frac{h}{l}$ and $\cos\theta = \frac{b}{l}$; therefore

$$P = W\frac{h}{l} \text{ and } R = W\frac{b}{l}.$$

Or the triangle CAB can be taken as the triangle to forces, for its sides can be made to be parallel to the three forces simply by turning it round and over so that its sides, h, b, l, are parallel to the forces P, R, and W respectively; and are therefore proportional to them;

$$\text{or } \frac{P}{h} = \frac{R}{b} = \frac{W}{l}.$$

3. The similar problem but with P horizontal (Fig. 27).

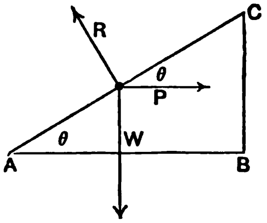


Fig. 27.

In this case it is simplest to resolve vertically and parallel to the plan. We get thereby

$$R\cos\theta = W.$$
$$P\cos\theta - W\sin\theta = 0.$$
$$\text{or } R = \frac{W}{\cos\theta} \text{ and } P = W\tan\theta,$$

We have deduced the condition of equilibrium from the laws of motion, but it can be experimentally verified—at any rate with a fair amount of accuracy—by means of strings and pulleys. Three pulley wheels are mounted as shown in Fig. 28; three strings knotted together at one point pass over these pulleys and weights are then attached to them. The knot, which represents a particle, settles down, if possible, into a state of rest. The direction of the strings can be transferred to a sheet of paper beneath them by means of a set square, and the angles between them are then accurately measured. Their sines should be proportional to the weights attached to the strings whose direction is represented by the sides opposite them. Or a triangle is drawn with its sides parallel to the three strings and the lengths of the sides compared with the forces parallel to them.

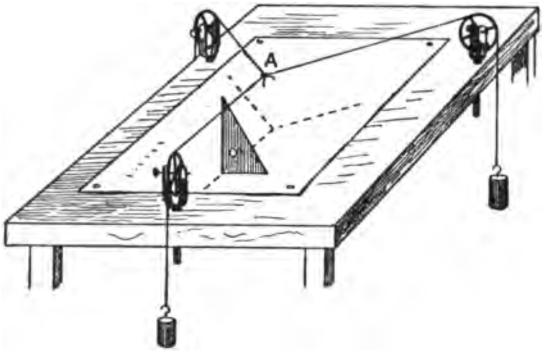


Fig. 28.

Another mode of illustrating the law, and one which has the advantage that no pulleys, with their accompanying friction, are employed, is known as the cord polygon. A long thread attached to points A and B of a beam has other threads knotted to it at various points, and weights are hung by these. Fig. 29 shows three of these strings with weights. (Those hung from A and B are only plumb lines.) Each knot is in equilibrium under the action of the tensions in the three portions of string meeting at it. For example, at the left-hand knot the forces are T_1, T_2, and the weight w_1. If a triangle be drawn with its sides parallel to these forces they will also be proportional to them. Such a triangle is *ab*O (Fig. 30). Again at the second knot the forces are T_2 (equal but opposite to that in the first, and therefore represented by $\overline{Ob}$), T_3, and w_2. If $\overline{ab}$ be continued so that $\overline{bc}$ equals w_2 the tension T_3 must be parallel to $\overline{cO}$, for the other two forces are represented by $\overline{Ob}$ and $\overline{bc}$. Thus both triangles have a common apex, O. This statement is true for the triangle

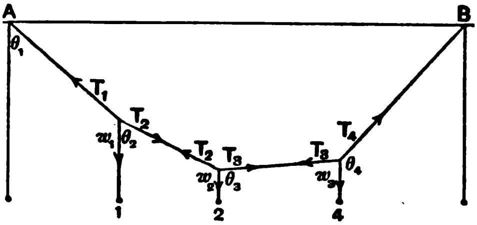


Fig. 29.

corresponding to *each* knot; hence we may say: If a vertical line $\overline{ad}$ be drawn and divided into lengths proportional to the weights w_1, w_2, w_3, and lines be drawn from a, b, c, d, parallel to the inclined strings; these will all meet in one point. Such a diagram is called a *force diagram.* In order to draw lines parallel to the strings, either the angles θ_1, θ_2, etc., can be measured with a protractor; or, better, measure the horizontal distance of each knot from one of the plumb lines and its vertical distance from the beam, and plot on a reduced scale the figure taken by the cord.

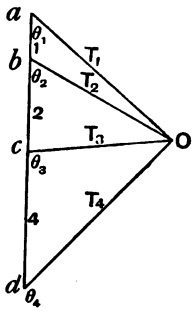


Fig. 30.

The algebraic expressions for the tensions are easily obtained by considering that *each* knot is in equilibrium. Hence we get—

$$T_1 \cos \theta_1 - T_2 \cos \theta_2 = w_1$$
$$T_2 \cos \theta_2 - T_3 \cos \theta_3 = w_2, \text{ etc.};$$

or, in words, the difference of the vertical components of the tensions in the inclined portions at each knot is equal to the weight suspended at it.

Again

$$T_1 \sin \theta_1 = T_2 \sin \theta_2 = T_3 \sin \theta_3 = \text{etc.};$$

or the horizontal components are all equal.

These results can also be obtained by inspection of the force diagram,

EXAMPLES TO CHAPTER VI.

1. Find the horizontal force which will maintain a particle of 100 grams mass in equilibrium on a smooth inclined plane whose angle is 30°.

2. What mass will this force keep in equilibrium if it acts parallel to the plane?

3. Three forces acting on a particle are represented by the straight lines drawn from each angle of a triangle to the middle of the opposite side; show that equilibrium results.

4. A mass of 120 grams suspended from a point by a string is pulled horizontally by a force of 10,000 dynes; find the inclination of the string to the vertical, when the forces are in equilibrium.

5. Three equal forces acting at a point are in equilibrium; show that the angle between any two of them must be 120°.

6. A smooth ring holding up a weight of 2 lb. is carried on a cord 120 cms. long, whose ends are attached to two points on the same horizontal line and 100 cms. apart. Find in lb. weight the tension in the cord.

7. Four forces keeping a particle in equilibrium are parallel to the sides of a square taken the same way round; show that they are not necessarily proportional to them.

8. A body of mass 100 lb. is kept at rest on a smooth plane of 45° angle by means of a string which passes over a smoothly mounted

pulley at the top of the plane, and then hangs vertically with a weight at its free end. What is the value of the weight?

9. If this weight is 100 lb. weight and rests on the ground, what is the pressure on the ground, other things being as in the last question.

CHAPTER VII

MOTION ON CURVES

UP to this stage we have considered only those cases in which the resultant force acts along the direction of motion, the result being that the body continues to move in the same straight line. The effect of the force is then merely to alter the velocity in this line. But if a force acts in a direction inclined to the motion, it produces also a component velocity at right angles thereto. And since the direction of motion is that of the *resultant* velocity this direction will no longer remain the same. For example, let the line $\overline{AB}$ (Fig. 31) represent the velocity at a particular instant. Let the line $\overline{BC}$ be the direction of the force acting, and therefore of the acceleration also. The velocity gained in consequence of this acceleration in any very short interval t is at—represented by $\overline{BC}$; the resultant velocity is $\overline{AC}$ (or V^1). Thus the velocity has changed both in

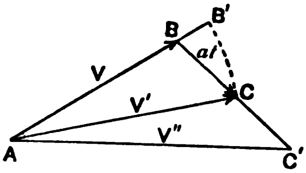


Fig. 31.

direction and magnitude. If the force remains constant in direction and magnitude, the velocity at the end of another equal time interval is V^{11}, where $\overline{CC^1}$ is made equal to $\overline{BC}$; hence both direction and magnitude of the velocity have again changed. If the change were the result of equal *blows* delivered in the direction $\overline{BC}$, the velocity would change by jumps; but a continuous force produces a continuous change. If we select for examination an earlier time than t, V^1 will have a magnitude and direction intermediate between $\overline{AB}$ and $\overline{AC}$; in fact, as time progresses from the instant at which the velocity is $\overline{AB}$ it takes all possible values and directions between $\overline{AB}$ and $\overline{AC}$. Now the path has the same direction at any instant as the resultant velocity at the same instant; hence the path changes from the direction $\overline{AB}$ to the direction $\overline{AC}$, and does so continuously; in other words the path is curved. If the force is not constant $\overline{CC^1}$ may neither be the same length nor in the same direction as $\overline{BC}$; the whole line between B and C^1 will then be curved; but the same construction applies.

If an arc CB^1 be drawn with A as centre, $\overline{BB^1}$ represents the increase in the numerical value of V; now $\overline{BC}$ $(= at)$ is the resultant of the two components $\overline{BB^1}$ and $\overline{B^1C}$; hence $\overline{B^1C}$ (which becomes straight and at right angles to V when t is chosen indefinitely small) is the component to which the deviation of V is due. If the force (and therefore $\overline{BC}$) is wholly at right angles to V, its effect is only to deviate the velocity without changing the magnitude; and if the force changes direction continuously, so as always to be at right angles to the velocity, the latter will be permanently of constant magnitude but of changing direction. The simplest case is that in which the force has also a constant magnitude. We now consider this in detail.

1. To show that if a particle moves with constant speed in a circle, it must be subject to a force of constant magnitude always directed towards the centre of the circle; and to find this force.

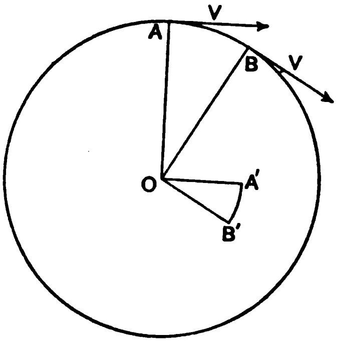


Fig. 32.

The velocity is of constant magnitude, hence the acceleration and force are at any instant wholly at right angles to the path at the same instant; the line of direction always, thus, passes through the centre. To find the magnitude, we select any short length $\overline{AB}$ of the path. From the centre O draw equal lines parallel to the tangents at A and B; these lines, $\overline{OA^1}$ and $\overline{OB^1}$, will represent the velocities at A and B. A curve such as A^1B^1 drawn through the ends of all such lines is called a *hodograph*; the angle A^1OB^1 is obviously equal to the angle AOB. Now OB^1 is the resultant of OA^1 and A^1B^1. Therefore $\overline{A^1B^1}$ which, when the angle at O is indefinitely small, may be considered straight, is the velocity gained while $\overline{AB}$ is passed over—*i.e.* it is at where a is the acceleration.

$$\text{Now, } \overline{A^1B^1} : \overline{OA^1} :: \overline{AB} : \overline{AO},$$

$$\text{or} \quad \frac{at}{V} = \frac{Vt}{R}, \quad \therefore a = \frac{V^2}{R}.$$

The velocity, V, may be expressed in terms of the rate at which the angle AOB increases as the point B moves round the circle; this rate of increase is the *angular velocity* with which a radius $\overline{OB}$ attached to the particle turns round; it is commonly spoken of as the angular velocity of the particle. Denoting it when expressed in radians per second by ω, then $V = \omega R$, and therefore $a = \omega^2 . R$. If the stone goes round n times per second, $\omega = 2\pi n$. Hence $a = 4\pi^2 n^2 R$.

The acceleration a is called a *central* acceleration, because its direction always passes through a fixed centre.

A central acceleration implies a central force whose value is the mass, m, multiplied into the acceleration. Hence the central force in this case is equal to $\frac{mV^2}{R}$ or $m\omega^2 R$. Since the acceleration and force both point *toward* the centre, they are known as *centripetal.*

When a stone is whirled round by a string in a horizontal plane, the conditions are very nearly those here discussed. The centripetal force on the stone is the tension in the string.

The motion of the moon is also very nearly constant in a circular path. The centripetal force on the moon is the earth's gravitational force.

It should be noticed particularly that there is in these cases no force acting upon the body except the centripetal force, whose value we have found. In the early history of the subject it was thought to be convenient to regard the body as being in equilibrium, and it was considered as possessing another force, which, together with the centripetal one, maintained equilibrium. Since, in order to do so, this hypothetical force must point *outwards from* the centre, it was called the "centrifugal force."

With regard to this question, all we need do is to point out that as there is acceleration, the body is certainly *not* in equilibrium, and it can only tend toward confusion of ideas to consider it as being so; the notion will therefore not be entertained here.

In order to continue to use the term, it has been sought to apply it to the reaction on the point of support at the centre (*i.e.* the reaction of the stone on the hand, or of the moon on the earth, etc.). This reaction must, of course, point outwards, and it is a real force according to our definition; the name would therefore be quite applicable. But, to avoid ambiguity, it is best to drop the term altogether.

Example: If a stone of 1 lb. mass is whirled round at the end of a string 2 ft. long, making one revolution per second, the tension in the string must be

$$\frac{W}{g} \times 4\pi^2 \times 2 \bumpeq \frac{80}{32} \text{ lb. wt.} = 2\tfrac{1}{2} \text{ lb. wt.}$$

If the string is not able to exert this tension it *breaks*, and the stone flies off along a *tangent* to the circle; simply because the centripetal force has ceased to act, and it must then move in a straight line unless other forces come into play. There is no evidence here of a force tending away from the centre, but rather of the absence of any such force.

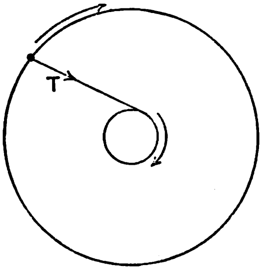


Fig. 33.

While the stone is acquiring its speed in the circle there must be also a component force tangential to the circle. In practice this is supplied by the stone lagging behind the hand, so that the string is never at right angles to the motion. The hand itself moves round a small circle as in Fig. 33.

Since the earth rotates uniformly on its axis, any body resting on it is moving round in a circle with constant speed; it therefore has a centripetal acceleration, and this must be produced at the expense of the weight of the body; that is to say, the body will press downwards with less force than its weight upon the earth which supports it. For example, let the body be on the equator, and let v be the velocity. The forces on it are its weight, $W = mg$ downwards, and the pressure of the earth upwards. The resultant force downwards is $W - P$, and this is the measure of the centripetal force whose value is also given by mv^2/R, where R is the radius of the earth. Hence,

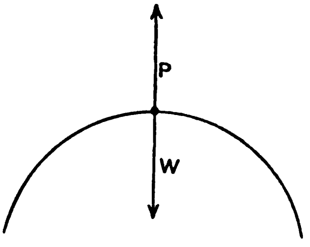


Fig. 34.

$$W - P = mv^2/R$$

$$\text{or} \quad P = m\left(g - \frac{v^2}{R}\right).$$

Now P is the *apparent* weight of the body—the weight, that is to say, which a spring balance indicates. The real weight is here taken as mg. But it must be carefully noticed that observations of g in any locality only give the acceleration $g - \frac{v^2}{R}$ relative to the surface, and not the true value measured with respect to the centre of the earth. Hence part of the variation with locality of the observed values of g arises from the difference of the velocity of the surface of the earth in different latitudes.

Taking the radius of the earth as $6{\cdot}38 \times 10^8$ cms., the value of v^2/R at the equator is

$$6{\cdot}38 \times 10^8\left(\frac{2\pi}{86164}\right)^2 = 3{\cdot}39\,\frac{\text{cms.}}{\text{sec.}^2}$$

and this is about 1/289 part of g. The true value of the acceleration of a falling body with respect to the centre of the earth is about 3·39 cms. per sec. per sec. greater than its value measured with respect to the surface at the point of the equator to which it falls.

When the point of observation is in latitude λ, the radius of the path is reduced in the ratio of 1 to $\cos\lambda$; and moreover it is only the normal component which affects the normal force. Hence the diminution of true gravity in latitude λ is $3·39 \cos^2\lambda$ cms. per sec. per sec.

If the earth were revolving about seventeen times as fast as at present, then, since $17^2 = 289$ the value of the observed acceleration at the equator would be zero. The slightest force would lift a body from the ground or cause it to return there. The weight at the poles, since $\cos 90°$ is zero, is not affected by the rotation.

Rail Elevation.—When a train is going round a curve, the centripetal force causing it to do so must be supplied by the rail. Since the flanges of the wheels are on the inside it is the outer rail which must exert the required force. To relieve the strain from the flange, the outer rail is raised a little, so that the force normal to the upper surface of the rails has a horizontal component. Let h be the elevation of the outer rail, and b the distance between the rails; then the horizontal component of the forces on both rails is $W\frac{h}{b}$; if this be equated to $Wv^2/(Rg)$, where R is the radius of curvature of the line and v is an average velocity—say thirty miles an hour—the value that h should have is determined; the equation gives

$$h = \frac{v^2 b}{Rg}.$$

Example: If $b = 4$ ft., $v = 30$ miles per hour, and R = half a mile,

$$h = \frac{2 \times 30^2 \times 5280 \times 4}{(3600)^2 \times 32} = \frac{11}{120} \text{ feet.}$$

Conical Pendulum. If a small bob be suspended by a string from a support, and the support is rotated about a vertical axis, the string remains vertical until a certain angular velocity is reached; the bob then moves outwards from the centre, and continues to rotate in a circle of constant radius if the angular velocity is maintained constant.

Assume that the string makes a constant angle, θ, with the vertical; let l be its length, and ω its angular velocity round the axis, $\overline{OO'}$. The forces on the bob are the tension, T, in the string and its weight, W.

Since there is no upward acceleration, $T \cos \theta = W$. The horizontal component of T, *viz.*, $T \sin \theta$, produces the centripetal acceleration $\omega^2 . r$, conse-

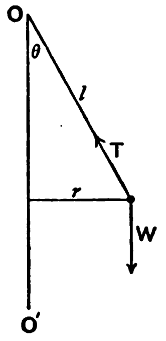


Fig. 35.

quently $T \sin \theta = m\omega^2 . r$, or putting $T \cos \theta = mg$, and $r = l \sin \theta$,

$$\cos \theta = \frac{g}{l\omega^2}.$$

If $\omega^2 = g/l$, $\cos \theta = 1$, and therefore θ is zero; for greater values of ω, θ has a finite value; for less values, the bob and string spin round the axis $\overline{OO'}$ with which they coincide. Indeed, this is obviously a possible kind of motion, whatever the angular velocity may be; the fact is, however, that it becomes unstable at the critical value obtained above; and, at higher values of ω, any minute disturbance, which causes the string to deviate never so slightly from the vertical, sets in play an *outward* force which causes the displacement to increase. The time of a complete revolution (or *time period*) is the time required to traverse the angle 2π radians with an angular velocity ω, *i.e.*

$$2\pi/\omega \text{ or } 2\pi \sqrt{\frac{l \cos \theta}{g}}.$$

2. **Second Case of Motion on a Curve.** A small body is projected obliquely, and is subject only to gravity during the motion; discuss the motion.

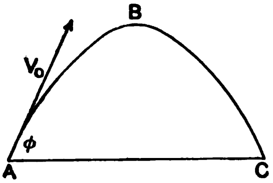


Fig. 36.

This is the case of a projectile; at least, in so far as the resistance of the air can be neglected in the actual case. Let its initial velocity be V_0, making an angle ϕ with the horizontal; this angle also denotes the initial direction of its path, and is called the *elevation.*

Draw a line OA as in Fig. 37 to represent V_0; from A drop a vertical; then the acceleration g is always along this vertical, which is therefore the hodograph. Equal lengths on this vertical represent equal increments of vertical velocity, and correspond to equal times; thus V_0, V_1, V_2, V_3, etc., represent the resultant velocities at equal intervals, t.

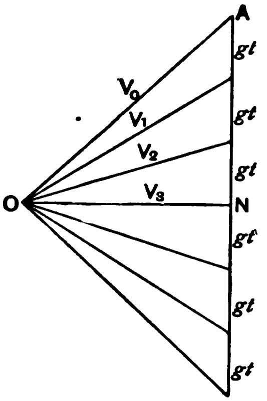


Fig. 37.

The horizontal component of each of these is ON = $V_0 \cos \phi = v_0$ (say). The constancy of this component, of course, follows from the absence of any horizontal component of acceleration. Similarly the vertical velocity at any instant is $u_0 - gt$ where $u_0 = V_0 \sin \phi$—*i.e,* the initial vertical velocity. Let the vertical and horizontal displacements be represented by h and x; then

$$(1) \quad x = vt = v_0 t$$

$$(2) \quad h = u_0 t - \tfrac{1}{2}gt^2$$

$$= \frac{u_0}{v_0}x - \tfrac{1}{2}g\frac{x^2}{v_0^2}.$$

When h is O, let $x = X$

$$\text{Then}\quad \tfrac{1}{2}\frac{gX^2}{v_0^2} = \frac{u_0 X}{v_0}.$$

$$\therefore\ X \text{ either} = O \text{ (\textit{i.e.} the starting point);}$$

$$\text{or else} = \frac{2u_0v_0}{g} = \frac{2V^2 \sin\phi \cos\phi}{g} = \frac{V^2 \sin 2\phi}{g}.$$

This value of X is called the horizontal range (H.R.).

Since $\sin 2\phi = \sin(180° - 2\phi)$, the horizontal range is the same for ϕ as for $90° - \phi$, assuming of course that V is the same in both cases—*i.e.*, if the elevations in the two cases add up to 90°, the horizontal range is the same. The H.R. is the greatest possible for a given value of V when $\sin 2\phi = 1$: *i.e.*, when $\phi = 45°$; its value is then $\frac{V^2}{g}$.

Since the vertical component of velocity is reduced to zero in the time u_0/g, the path is then horizontal; and since the vertical component afterwards becomes downward, this is the summit or highest point of the path; the height, H, of the summit is $\frac{u_0^2}{2g}$. Since it takes as long to traverse the vertical displacement downwards as upwards, the time taken to traverse the whole distance from ground to ground again is equal to u_0^2/g; this is called the *time of flight.*

The shape of the path could be obtained by plotting h as given in Equation 2 against x. Everything is constant in this equation except *h* and *x*. The curve which will be obtained is called a *parabola*. The equation can be simplified by transforming it.

$$\text{Let } y = \mathrm{H} - h = \frac{u_0^2}{2g} - h,$$

$$\text{and } w = x - \tfrac{1}{2}\mathrm{X} = x - \frac{u_0 v_0}{g},$$

then Equation 2 becomes

$$\frac{u_0^2}{2g} - y = \frac{u_0}{v_0}\left(w + \frac{u_0 v_0}{g}\right) - \frac{g}{2}\frac{\left(w + \frac{u_0 v_0}{g}\right)^2}{v_0^2}$$

$$= \frac{u_0 w}{v_0} + \frac{u_0^2}{g} - \frac{g w^2}{2 v_0^2} - \frac{u_0 w}{v_0} - \frac{u_0^2}{2g},$$

$$\text{or } + y = \frac{g w^2}{2 v_0^2} = k w^2.$$

Thus we see that *y* is proportional to the square of *w*. The transformation effected above changes the *origin* of the curve from the starting point to the summit, and turns the curve upside down, but makes no other change. The value of *y* must be the same for equal positive and negative values of *w*—*i.e.* it is symmetrical with respect to a vertical line through the summit.

If the ground is not horizontal it is best to resolve the motion parallel and normally to it. Thus, if the inclination of the ground is θ, and ϕ is the elevation measured with respect to the ground, the components of the initial velocity V_0 are V_0

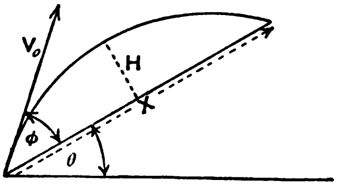


Fig. 38.

$\sin\phi$ (normal) and $V_0 \cos\phi$ (parallel). The corresponding components of the acceleration g are $g\cos\theta$ (normal) and $g\sin\theta$ (parallel). Hence the time required to reach the greatest distance H from the plane is $\frac{V_0 \sin\phi}{g\cos\theta}$, and the greatest distance itself is $\frac{V_0^2 \sin^2\phi}{2g\cos\theta}$. The equation giving any distance h from the plane is

$$h = V_0 \sin\phi \, . \, t - \tfrac{1}{2}g\cos\theta \, . \, t^2$$

$$\text{and} \quad x = V_0 \cos\phi \, . \, t - \tfrac{1}{2}g\sin\theta \, . \, t^2,$$

where x is measured up along it from the starting point.

When $h = o$, $t = \frac{2V_0 \sin\phi}{g\cos\theta}$, which is therefore the time of flight, and is also twice the time required to attain the greatest distance, H, from the plane. This might have been inferred from the fact that at the greatest distance the velocity perpendicular to the plane has been reduced to zero. The normal velocity regained, with opposite sign, in falling the distance H under uniform acceleration, is the same as the initial normal velocity, and the time taken in acquiring it is equal to the time taken in losing it. It must not be inferred, however, that the curve is symmetrical with regard to H, as in the former case; the reason it is not so is that the parallel component of velocity diminishes throughout the journey instead of remaining constant. The path is of course the same as if the inclined plane were not present, and the initial elevation with respect to the horizontal ground were $\theta + \phi$.

The range X measured along the plane is the value of x corresponding to $t = 2V_0 \sin \phi/(g \cos \theta)$ that is,

$$X = V_0 \cos \phi \,.\, \frac{2V_0 \sin \phi}{g \cos \theta} - \tfrac{1}{2} g \sin \theta \,.\, \frac{4V_0^2 \sin^2 \phi}{g^2 \cos^2 \theta}$$

$$= \frac{2V_0^2 \sin \phi}{g \cos^2 \theta}(\cos \phi \cos \theta - \sin \phi \sin \theta)$$

$$= \frac{2V_0^2 \sin \phi}{g \cos^2 \theta} \cos(\phi + \theta).$$

These results only apply if the resistance of the air is of negligible influence. This is approximately so for slow velocities and dense round projectiles which have large mass for small surface. Complications that can arise in other cases are well illustrated by the eccentricities of motion of a golf ball, or of a toy balloon, or a piece of card. These cases lie outside the scope of this book.

EXAMPLES TO CHAPTER VII.

1. A stone is whirled round 20 times a minute at the end of a string 50 cm. long, its path being horizontal; find the centripetal acceleration and the centripetal force if the mass is 100 grams.

2. If it be whirled round in a vertical plane, at which point of the motion will the string be most likely to break?

3. Show that, assuming that the pull of the

earth varies inversely as the square of the distance, it is just sufficient at the distance of the moon to keep it in its circular path.

Lunar month = $27\frac{1}{3}$ days
Distance of moon = 60 times the radius of earth.

4. A train of 400 tons weight goes round a curve whose radius of curvature is 200 yds. at the rate of 30 miles an hour. The rails being 4 ft. apart, what should be the elevation of the outer rail?

5. If the outer rail be adjusted for a speed of 30 miles an hour, what will be the force on the flange, if the velocity is 45 miles per hour, or if it is only 15?

6. Show that, if several strings of different lengths carrying weights be suspended all from the same point, and spin round as conical pendulums with the same angular velocity, the weights will lie in a horizontal plane.

7. If the point of suspension of a conical pendulum be distant never so little from the axis of rotation, show that it will become deviated, however small the angular velocity is.

8. Calculate the angular deviation of a conical pendulum of length 25 cm. when rotating twice a second.

CHAPTER VIII

WORK AND ENERGY

ALTHOUGH the quantities already referred to are sufficient for dealing with purely mechanical phenomena, it is convenient to introduce others which are of special value in particular cases. The conceptions whose names appear at the head of this chapter have proved to be exceedingly useful especially in those cases which involve also other branches of Physics (Heat, Electricity, etc.). In the present chapter we shall only deal with the simplest examples.

Definition of Work. When a body moves in the direction of a force the latter is said to do *work*; the amount done is measured by the product of the force into the distance moved in its direction. Thus when force and motion have the same direction (Fig. 39), the work w is

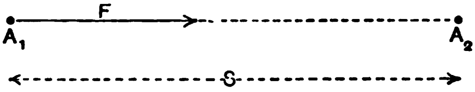


Fig. 39.

$$w = Fs,$$

s being the displacement. When the directions are not the same the displacement in the direction

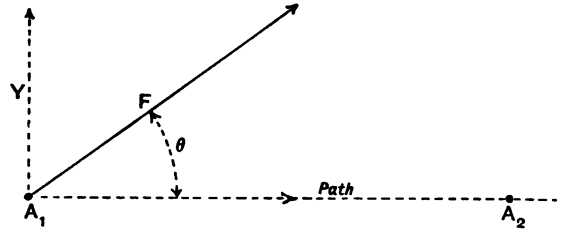


Fig. 40.

of F is $s \cos \theta$ (Fig. 40), and the work done is $w = Fs \cos \theta$ where $s = A_1A_2$. Since this may be written $F \cos \theta . s$ we may define the work done as being equal to the product of the displacement s into the component force in its direction.

Units of Work. 1. International unit. When F is one dyne, and s is one centimetre, the work done is said to be one dyne-centimetre or *one erg*. Thus, when a mass of one gram falls, the work done by gravity for each centimetre is 981 ergs.

2. British gravitational unit. When F is one pound weight and s is one foot, the work done is said to be one *foot-pound*. Thus, when one pound falls through one foot, gravity does work equal to one foot-pound. If it moves horizontally no work is done by gravity; though, of course, other forces may be doing work. If the motion is in the opposite sense to that of the force

the latter is said to do *negative* work, or the body is said to do positive work *against* the force.

Both force and motion are necessary in order that work may be done. No work is done by the force between a pillar and the floor it supports (unless subsidence is taking place). Again no work would be done by a body moving with constant velocity in a straight line, for there would then be no force acting on the body.

Theorem. When several forces are acting on a body the sum of the amounts of work done by them is equal to the work that would be done by their resultant acting alone.

Thus let several forces act as in Fig. 41, and let the actual motion of the body be from

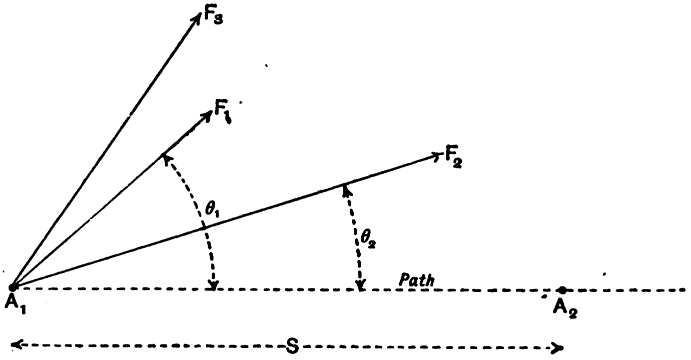


Fig. 41.

A_1 to A_2 a distance s. The sum of the amounts of work done is

$$(F_1 \cos \theta_1 + F_2 \cos \theta_2 + F_3 \cos \theta_3)s.$$

But this equals Xs (where X is the component in the direction A_1A_2 of the resultant) and is therefore equal to the work that would be done by the resultant.

So far we have tacitly supposed the force to be constant and the path to be straight. The necessary extension if these restrictions do not hold is that the whole journey must be divided into very short steps, so short that each may be considered straight, and the force to be sensibly constant while it is traversed. The work done in each step is then calculated as before, and the algebraic sum of the separate values taken. The *limiting value of this sum* as the steps are made more and more numerous is the work done along the whole path.

Energy. The conditions under which a body in translatory motion may do positive work *against* a force are twofold :—

1. It can do such work, if no other force is acting, if it initially possesses a velocity in the opposite direction to that of the force ; for it will then continue to move in opposition to the force until its velocity is reduced to zero.

2. When it is acted upon by a second force of opposite sense to and of greater magnitude than the given one. For the acceleration will be in the sense of the greater force ; thus the motion produced will be in opposition to the given force, and work will be done against it.

In all cases when a body can do work against a force it is said to possess ***energy***; and the energy is measured so that its decrease is equal to the work done. When the energy arises from motion it is called ***kinetic***, when from the relative positions of bodies it is ***potential*** or ***positional.*** We will now derive expressions by which the energy can be calculated in different cases.

1. *Kinetic energy.* Let F be the value of the only force acting. Then we know that

$$v_2^2 - v_1^2 = 2as = \frac{2\mathrm{F}s}{m},$$

$$\text{or}\quad \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_2^2 = -\,\mathrm{F}s,$$

where v_1 is the velocity at any time, and v_2 the velocity after the body has moved through a distance s. Now the left-hand side of this equation is the decrease (*i.e.* the initial minus the final value) of the quantity $\frac{1}{2}mv^2$; and $-\,\mathrm{F}s$ is the work done *against* the force F. The latter term is positive if F is negative—*i.e.* if it is an opposing force; and the quantity $\frac{1}{2}mv^2$ must then have undergone an equal positive decrease. Now, positive work against F will continue to be done until v_2 has become zero; hence according to the definition the quantity $\frac{1}{2}mv^2$ is the kinetic energy when the velocity is v.

2. *Potential energy.* This may arise in several ways. The most important is the energy due to the relative position of the body and the earth which attracts it. A body raised from the ground and let fall can do work against

any opposing force which is less than the weight of the body. Hence the greatest work it can do in a fall equal to s is Ws or mgs. When it reaches the ground it can—under usual circumstances—fall no further. If h is the initial height above the ground, the maximum possible work that can be done against an opposing force is Wh or mgh. This is taken as a measure of its potential energy in the initial position. If the opposing force is less than W, gravity still does work equal to Ws during a fall, s; but in this case the motion takes place with acceleration, *i.e.* the body gains velocity downwards and therefore gains kinetic energy. This can be seen clearly from the following equation :

$$v_2^2 - v_1^2 = 2as,$$

$$\therefore \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = mas = (\mathrm{W} + \mathrm{F})s = \mathrm{W}s + \mathrm{F}s,$$

which expressed in words states that—

$$\left.\begin{array}{c}\text{Increase of}\\ \text{kinetic}\\ \text{energy.}\end{array}\right\} = \left\{\begin{array}{l}\text{Decrease of potential energy of}\\ \text{gravity } \textit{minus} \text{ work done } \textit{against}\\ \text{the force F.}\end{array}\right.$$

We shall later on examine this complete equation more fully. In the present place where introductory notions only are dealt with we shall suppose no force but gravity present. In such a case, that of a freely falling stone for example, the energy changes from the potential to the kinetic form during the fall, an increase of one kind corresponding to an equal decrease of the other. For example, if the falling body has a velocity v_1 when at a height

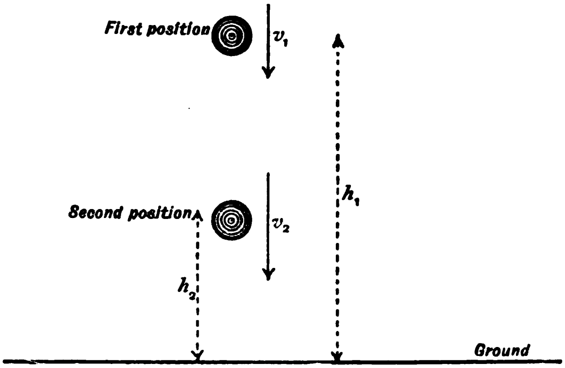


Fig. 42.

h_1 (Fig. 42) and a velocity v_2 when at a height h_2 we have the relation

$$\begin{aligned}\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 &= \text{decrease of potential energy}\\ &= mg(h_1 - h_2).\end{aligned}$$

Again, if the motion is upward the potential energy increases, the kinetic energy diminishing by an equal amount. Thus, whether the motion is up or down, the total quantity of both kinds remains constant.

In the above discussion the motion has been assumed to take place in a vertical straight line. But now comes an important question: How will the potential energy change if the motion takes place between the same two points but not in a straight line?

The answer to this question is that the change of potential energy is independent of the path by which the body passes from one point to another. We prove this important theorem in two stages.

1. *Work done by gravity during inclined motion.*

Let the motion be along the straight line PQ making an angle θ with the vertical. The work done by gravity is W cos $\theta . \overline{PQ}$ or $W . \overline{PQ} . \cos\theta = W . \overline{PP}^1$. Or in words, the work done by gravity when a body falls along an inclined path is equal to the work done if it falls vertically through the same vertical height. If the body be now moved from Q to P′ no work is done by gravity, because this motion is wholly horizontal. Hence the total work done in moving along the path PQP′ is the same as in moving direct along PP′.

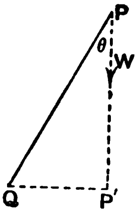


Fig. 43.

Next consider what becomes of it. If PQ were a smooth inclined plane the acceleration down the plane is $\frac{W\cos\theta}{m}$ or $g\cos\theta$; therefore, if v be the velocity measured down the plane,

$$v_2^2 - v_1^2 = 2g\cos\theta . \overline{PQ},$$

$$\text{or } \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = mg\cos\theta . PQ = mg . PP'.$$

Thus the work done by gravity is again converted into kinetic energy.

We took the case of a smooth plane in order that gravity might be the only force doing work. The pressure due to the plane in this case does none, for the motion is wholly at right angles to it.

2. *Work done by gravity during curved motion.* Let the motion be any curved path from P to P′ (Fig. 44). Select any very short length s of the path. If the length is so short that it may be considered straight the work done by gravity while the body moves along it is $W\overline{pq}$ where $\overline{pq}$ represents the vertical distance corresponding to s. Hence the total work done between P and P′ is $\Sigma(W.\overline{pq})$ or since W is a constant $W\,\Sigma\overline{pq}$; that is $W.\overline{PP'}$. But this is the work done in falling the vertical height $\overline{PP'}$; thus the work done by gravity between P and P′ is independent of the path taken. If the curved path be considered as a succession of smooth planes the total work done is again seen to be transformed into kinetic energy. It is this property which makes it useful to speak of the potential energy at a point. It is unnecessary to specify anything with regard to the direction in which the body will move from it. The quantity of work which gravity does on a body of given mass, or, as we shall say, the decrement of potential energy which takes place as the body of given mass moves from one point to another depends upon the two points, and, as far as we know, upon no other circumstances whatever. If there are

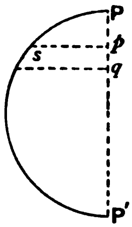


Fig. 44.

no other forces having a component along the line of motion—so that no work is done except by gravity—the decrement of potential energy is equal to the increment of kinetic energy.

This is the doctrine of the Conservation of Energy. The full elucidation of the law must be reserved to a later section (Chap. XV).

As a further illustration consider the case of a projectile thrown obliquely. The foregoing principle teaches that if the ground is horizontal the velocity on reaching it must be numerically equal to the velocity on leaving it.

The summit being at a height, h, and the velocity of the particle there being V

$$\tfrac{1}{2}m(V_0^2 - V^2) = mgh,$$
$$\text{or, } V^2 = V_0^2 - 2gh.$$

At any point of the upward journey, the velocity must be numerically the same as at the same level on the downward journey, because the potential energy does not change on the whole. The student should verify from the considerations of the previous chapter, and without making use of the principle of Energy, that these results are true.

Another important application of the principle of energy is to find the law according to which a pendulum bob moves. A simple pendulum consists of an indefinitely small body hung by a string from a fixed point. If it is drawn aside through an angle a and then let go, what will be the velocity of the bob when the string makes an angle θ with the vertical? The forces acting on the bob are its weight, mg, and the tension T in the string. Since the motion is throughout at right angles to the string the tension does no work; the work done by gravity is $mg\,.\,\overline{H_1H_2}$, where $\overline{H_1H_2}$ is the vertical distance through which the bob has moved. There being no other forces present this work must equal the gain in kinetic energy. Hence the initial velocity being zero, and the velocity at the position θ being v we have

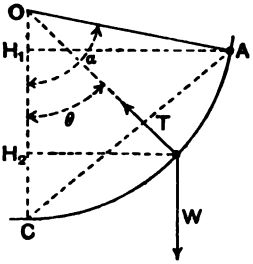


Fig. 45.

$$\tfrac{1}{2}mv^2 = mg\,\overline{H_1H_2} = mgl(\cos\theta - \cos\alpha),$$

$$\text{or } v^2 = 2gl(\cos\theta - \cos\alpha).$$

In particular if θ is zero, $\cos\theta = 1$,

$$\text{and } v^2 = 2gl\,(1 - \cos\alpha) = 4gl\sin^2\frac{\alpha}{2},$$

$$\text{and } v = \sqrt{gl}\,.\,2\sin\frac{\alpha}{2}.$$

Now the chord $\overline{AC} = 2l\sin\frac{\alpha}{2}$; hence

$$v = \sqrt{\frac{g}{l}} \times \text{chord } \overline{AC}.$$

Thus, the velocity gained in falling to its lowest position is simply proportional to the *chord* of fall. These results are true, however large the initial angle may be, provided the string keeps tight. For example, if a is 90°,

$$v = \frac{2}{\sqrt{2}}\sqrt{gl} = \sqrt{2gl},$$

which might at once be inferred from the fact that the vertical movement is in this case equal to the length of the string.

When the initial arc is very small there is no sensible difference in length between the chord and the arc itself; the latter may therefore be substituted for the former in the equation. It is this result which was assumed in experiments with Hicks' Balance (p. 39).

Power.—The rate of performance of work is called *power* or *activity*; its average value is obtained by dividing the total work done in any time interval by this interval. As unit power we take such as corresponds to the performance of unit work in unit time. For example, it is one erg per second in the Absolute C.G.S. system, and one foot lb. per second in the British Gravitational System. Neither of these units is of a convenient magnitude in engineering practice, and consequently multiples of them are chosen instead. Ten million ergs per second is called a *watt*, and one thousand watts a *kilowatt*; these two units are very generally used in electrical engineering. Five hundred and fifty foot lb. per second, or what comes to the same thing, 33,000 foot lb. per minute is called a *horsepower.* The relations between these units is shown in the following scheme :—

$$1\text{ watt} = \frac{10^7\text{ ergs}}{\text{sec.}} = \frac{10^7\text{ dyne cms.}}{\text{sec.}} = \frac{10^7}{981}\,\frac{\text{grm. wt. cms.}}{\text{sec.}}$$

$$= \frac{10^7}{981}\,\frac{\text{lb.wt.}}{453{\cdot}6}\,\frac{\text{foot}}{30{\cdot}48\text{ sec.}} = \frac{10^7 \times 60}{981 \times 453{\cdot}6 \times 30{\cdot}48}\,\frac{\text{foot lb.}}{\text{min.}}$$

$$= 746\text{ horsepower.}$$

Since the relation between the watt and the horsepower involves the acceleration due to gravity, it is different in different parts of the earth. It is a horsepower (which depends on the lb. weight) which is variable, and it is thus disqualified as a unit for scientific measurements ; for engineering work its constancy is quite sufficient.

EXAMPLES TO CHAPTER VIII.

1. Calculate the work done in lifting a 10lb. weight vertically through 100 ft.

2. Calculate the work done in drawing a body of 10 lb. weight through 100 ft. on a smooth plane inclined at 30° to the horizontal.

3. Calculate in ergs the work done in lifting a kilogram through a height of 360 metres.

4. If each of the operations in the previous questions occupies 10 minutes, calculate the power in each case.

5. What is the gain of kinetic energy while a mass of 200 grams falls freely through 40 metres ?

6. A stone of 10 lb. mass is thrown vertically with a velocity of 160 feet per second. Calculate its kinetic energy at the beginning, at the highest point of its path, half way up, and at the mid-time of the interval during which it rises.

7. What is the potential energy at the same points as in question 6?

8. A shot of 1,000 lb. is shot from a gun of 80 tons with a velocity of 1,200 feet per second. Which will do more work in coming to rest, the gun or the shot, and in what proportion? Show that the proportion is independent of the velocities.

9. A shot is fired from a gun whose elevation is 30° with an initial velocity of 1,000 feet per second. If the mass of the shot is 4 ounces what will be its kinetic energy when it is at a height of 1,000 ft., 2,000 ft., and at its highest point respectively?

10. The mass of a train is 100 tons, and the resistance to its motion is 20 lb. weight per ton; if the horse power of the engine is 250, what is the highest speed at which it can pull the train on the level?

11. How many ergs are there in a foot-pound?

12. A ball of 300 lb. moving with a velocity of 100 ft. per second pierces a fixed iron plate and then moves on with a velocity of 30 ft. per second; what amount of work is expended in piercing the plate?

CHAPTER IX

RIGID DYNAMICS

CENTRES OF MASS AND THEIR MOTION

A BODY of finite size can not only move forward; it can also turn round like a cart wheel; moreover its size and shape can change. In these cases different parts of it will move with different velocities. Yet no matter how complicated the body may be, we can determine the motion, at any rate theoretically, by applying the principles already discussed to each minute part of it. In other words if we imagine the entire body to be built up of particles, acting upon one another, and acted upon by the external forces present, the motion of each particle so determined will be the actual motion of the part of the body which it represents. At the outset, in applying this method it is necessary to define and find the position of a particular point in the body known as its *centre of mass.*

Definition of Mass-moment. The product of a mass into its perpendicular distance from any plane is called its *moment about the plane.*

Definition of Centre of Mass. Imagine the whole mass of the body, located at a point chosen so that the mass moment would then be the same *about any plane whatever*, as the sum of the moments of the several masses in their actual positions, then that point is their *centre of mass.*

In this introductory treatise we shall suppose the particles to be all in one plane. The perpendicular distance from a plane perpendicular to the given plane is thus the same as the perpendicular from a line drawn in the plane.

Let m_1, m_2, m_3 be three particles in the plane of the paper, and Ox a line in that plane; let the distances of the particles from Ox be denoted by y_1, y_2, y_3, as shown. Then if Y is the distance of their centre of mass from the same line, by definition

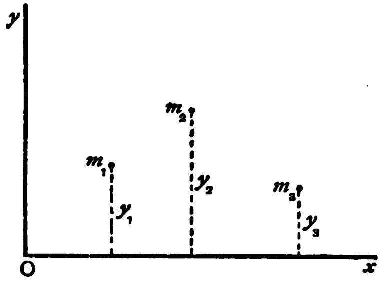


Fig. 46.

$$(m_1 + m_2 + m_3)\,Y = m_1 y_1 + m_2 y_2 + m_3 y_3.$$

If we find the distance from any other line such as Oy the centre is completely determined. It is obvious from the formula that the order of considering the particles has no influence on the result. It remains to prove that the choice of line of reference has no influence. For if we move Ox downwards parallel to itself through a distance a the distances are all increased by a; let Y′ be the new distance of centre of mass.

$$\text{Then} \quad (m_1 + m_2 + m_3)\,Y'$$
$$= m_1(y_1 + a) + m_2(y_2 + a) + m_3(y_3 + a)$$
$$= m_1y_1 + m_2y_2 + m_3y_3 + (m_1 + m_2 + m_3)a$$

$\therefore$ $Y' = Y + a$; which determines the same line as before.

Again, the same point is determined if the reference line is turned through any angle. We leave the proof of this—which is quite straightforward—to the student who has acquired some familiarity with Co-ordinate Geometry.

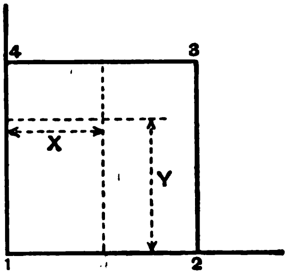


Fig. 47.

Example. Particles of masses 1, 2, 3, 4, are placed at the successive corners of a square whose side is a; find their centre of mass.

Take two sides as reference lines.

Then with the same notation

$$Y = \frac{4a + 3a + 1 \times 0 + 2 \times 0}{4 + 3 + 2 + 1} = \frac{7a}{10}$$

i.e. the centre of mass is on the horizontal dotted line as in Fig. 47.

Its distance from the vertical axis is

$$X = \frac{4 \times 0 + 1 \times 0 + 3 \times a + 2 \times a}{4 + 3 + 2 + 1} = \frac{a}{2}$$

i.e. half way across the square.

The student should take other lines of reference—for example, two bisectors parallel to the sides, or two diagonals—and show that the same point is then obtained. He must remember that distances on opposite sides of any line must be given opposite signs.

The following theorems are easily deduced and have frequent application:

If there are two particles only, their centre of mass is on the line joining them at a point which divides this line inversely as the masses.

If there are any number of particles distributed in a straight line, their centre of mass is also on this line.

In obtaining the centre of mass, the group may be divided into smaller groups whose mass-centres are first found; each small group may then be treated as a particle placed at its centre of mass; for the numerator and denominator of the expressions for Y and X are obviously unchanged by this modification of the method.

If the masses are distributed in the same way on the two sides of a line of geometric symmetry the centre of mass lies on that line. If there are two such lines it is at their point of intersection.

Thus, if equal masses are distributed uniformly over a square area, their centre of mass is at the intersection of the diagonals. A thin square plate (of uniform thickness) may be treated as such an assemblage. By the same rule the centre of mass is at the geometric centre in the case of a thin uniform straight wire; of a uniform circular or elliptical thin plate; and of a uniform wire bent in the form of a square, or circle, or ellipse. In the last three cases, it should be noted that the centre of mass is not in the substance of the wire itself.

To find the centre of mass of three equal masses, placed at the corners of a triangle.

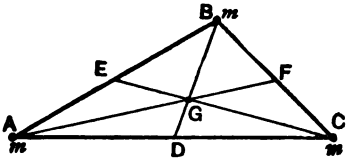


Fig. 48.

The centre of mass of A and C is midway between them at D; treat them as a single mass $= 2m$ at D. The centre of mass of it and B is $\frac{1}{3}$BD measured from D on the line DB, which is therefore the required point.

Note. If we had combined A and B into a double mass at E the required point is proved to be $\frac{1}{3}$EC from E; similarly by combining B and C it is shown to be $\frac{1}{3}$FA from F. But DB, EC, and FA are the three median lines of the triangle; hence these intersect at a point distant $\frac{1}{3}$ the length of each from the corresponding base.

To find the Centre of Mass of a thin uniform triangular Lamina.

Divide the lamina into very narrow strips parallel to AC; each of these may be treated as a line of particles whose centre of mass is at its mid-point. But the mid-points all lie on the median line BD of the triangle; hence the centre of mass of the triangle lies on this median. If we had divided the triangle into narrow strips parallel to AB, it is similarly proved to be on the median EC. It must therefore be at their point of intersection, the position of which we have found to be distant $\frac{1}{3}$ the length of each median from the corresponding base.

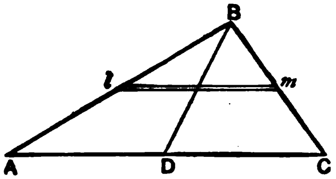


Fig. 49.

Cor. If (y_1, x_1) (y_2, x_2) (y_3, x_3) are rectangular co-ordinates of the corners of such a lamina, the co-ordinates of the centre of mass are

$$\frac{y_1 + y_2 + y_3}{3}, \frac{x_1 + x_2 + x_3}{3}.$$

A Wire bent into the form of a Triangle.

Replace the wire by three particles at the centres of the sides, and proportional to them respectively. Connect these par-

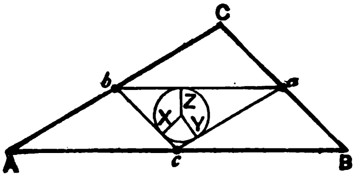


Fig. 50.

ticles and take the sides of this inner triangle successively as reference lines; let X Y Z be the distance of the centre of mass from them as shown (Fig. 50).

$$\text{Then} \quad X = \frac{a \times \frac{1}{2}\text{AC} \sin \widehat{bca}}{a+b+c} = \frac{\frac{1}{2} ab \sin \widehat{bca}}{a+b+c},$$

$$Y = \frac{b \times \frac{1}{2}\text{BC} \sin \widehat{bca}}{a+b+c} = \frac{\frac{1}{2} ab \sin \widehat{bca}}{a+b+c}.$$

Thus Y = X; similarly it is shown that Z = Y; hence, the centre of mass is the centre of a circle inscribed in the inner triangle.

A uniform wire bent into a circular arc subtending an angle 2β at centre of circle.

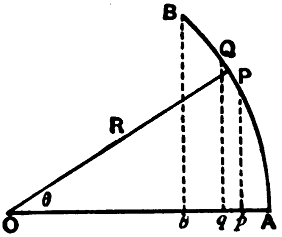


Fig. 51.

Let AB be the wire and O the centre of the arc of radius, R. Divide the wire into very short lengths such as $\overline{PQ}$; the distance of any such element from $\overline{OA}$ is R sin θ, and its mass is proportional to its length, hence if Y is the perpendicular distance of the centre of mass from $\overline{OA}$,

$$Y = \frac{\Sigma (R \sin \theta \,.\, \overline{PQ})}{\Sigma (\overline{PQ})},$$

$$= R \frac{\Sigma (\overline{PQ}) \,.\, \sin \theta)}{\Sigma (\overline{PQ})}, \text{ since R is a constant.}$$

Now $\overline{PQ} \sin \theta$ is the length of the projection $\overline{pq}$ of $\overline{PQ}$ on the line $\overline{OA}$; hence Σ ($\overline{PQ}$ sin θ) is the total projection $\overline{Ab}$; and $\Sigma\,(\overline{PQ})$ is the total length of the wire AB.

$$\therefore \qquad Y = \frac{R \times Ab}{AB} = \frac{R \sin^2 \beta}{\beta}.$$

Again from symmetry the centre of mass must be on the radius which bisects AB. Thus:

$$\overline{OG} = \frac{Y}{\sin \beta} = \frac{R \sin \beta}{\beta}.$$

For a quadrantal wire this becomes

$$\overline{OG} = \frac{R \sin \left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} = \frac{2R\sqrt{2}}{\pi}.$$

For a semi-circular wire

$$OG = \frac{R \sin \left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2R}{\pi}$$

Although we do not intend to consider in detail the case of solids, a few special rules and examples will now be given :—

If there is a plane of symmetry the centre of mass lies in that plane. If there are three it is at their point of intersection. Thus it is at the centre of a cube, parallelopiped, sphere, and ellipsoid.

Since a solid may be conceived as built up of a pile of plane laminæ, its centre of mass may be found by replacing each of these laminæ by a particle at its centre of mass and then finding the centre of mass of these equivalent particles.

Thus, the centre of mass of a tetrahedron is in the line joining the centre of mass of a lamina coinciding with the base to the opposite angle. Since each base may be considered in turn the point required must be the point of intersection of the lines so drawn from each angle. The position of the point may be found as follows :—

Consider equal particles placed at the corners of a tetrahedron. The centre of mass of any three of them is at the intersection of the median lines of the triangle they form ; the centre of mass of all four is, therefore, on the line joining this point with the opposite angle, and one-quarter the length of the line measured from the base. Since each three may be considered first in turn we reach the result that the lines drawn from the angular points to the centres of mass of laminæ, coinciding with the opposite bases, intersect at a point which is distant one-quarter the length of each of these lines from the corresponding base ; this is, therefore, also the centre of mass of a tetrahedron. It follows that the altitude (or *perpendicular* distance from the base) of the centre of mass is one quarter that of the opposite apex.

Pyramid with Base of any Number of Straight Sides.

Divide it into tetrahedra, having a common apex. The centre of mass of each has one-quarter the altitude of the pyramid, and therefore that of the pyramid has also. Again, it must lie on the line joining the centre of mass of a lamina coinciding with the base to the opposite corner.

Cone. A cone is the limiting case of a pyramid when the number of sides is made infinitely large.

Hollow Cone of uniform thickness. If the substance is thin it may be divided into triangular laminæ, having the apex of the cone as a common apex. The altitude of the centre of mass of each of these is one-third the altitude of the cone, and this must therefore be the altitude of the point required. Again, it must lie on the line joining the apex with the centre of mass of a ring coinciding with the periphery of the base.

If the substance is thick the centre of mass may be obtained from the fact that, together with a cone equal to the hollow inside, it would build up a solid cone having the same outside surface. Thus

Height of outside cone = h_1, area of its base A_1.
Height of inside conical hollow . . = h_2, area of its base A_2.
Mass of solid cone of height h_1 and base $A_1 = \frac{1}{3}\rho A_1 h_1$.
Mass of solid cone of height h_2 and base $A_2 = \frac{1}{3}\rho A_2 h_2$ where ρ is the density.

Then if Y is altitude of required centre of mass,

$$\tfrac{1}{4}h_1 = \frac{\tfrac{1}{3}\rho(A_1h_1 - A_2h_2)Y + \tfrac{1}{3}\rho A_2h_2 \times \dfrac{h_2}{4}}{\tfrac{1}{3}\rho A_1h_1}.$$

$$\text{Whence} \quad Y = \frac{A_1h_1^2 - A_2h_2^2}{4(A_1h_1 - A_2h_2)}.$$

Hemispherical Shell—*i.e.* **Hollow Hemisphere made of thin Uniform Material.** Draw a radius $\overline{Oy}$ perpendicular to base. Since this is an axis of symmetry it contains the centre of mass. We require then only to find the height $Y = \overline{OG}$. Slice the hemisphere into strips such as PPQQ parallel to the base. The mass of such a strip is proportional to its area. Its distance from the base is R sin θ. Hence

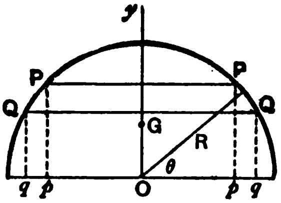


Fig. 52.

$$Y = \frac{\Sigma\,(R \sin\theta \times \text{area of strip})}{\Sigma\,(\text{area of strips})}.$$

But area of strip × sin θ is the area of a zone, pq, obtained by projecting the strip on the base, the sum of the areas of these zones is πR^2, and the sum of the areas of the strips is $2\pi R^2$. Since R is constant it may be transferred to the outside of the summation sign. Hence

$$Y = R \times \frac{\pi R^2}{2\pi R^2} = \frac{R}{2}.$$

This gives the position of the centre of mass for the curved part alone.

Solid Hemisphere. This may be treated as an assemblage of concentric hemispherical shells of equal thickness. The mass of each is proportional to r^2, where r is the radius of the particular shell; this may be considered as accumulated at its centre of mass at a distance $r/2$ from the base along the line of symmetry. Hence the position required is the same as that for a line of equally spaced particles, the mass of any particle being proportional to the square of its distance from one end of the row. The law of increase is thus the same as for masses of equally thick laminæ, of which a pyramid may be built, the height of this pyramid being half the radius. The centre of mass required is therefore three-quarters of the half radius from the base, *i.e.* three-eighths of the radius.

The above examples will illustrate how the position of this important point can be ascertained. Many more such examples could be given, although the solution of the problem requires, in general, a knowledge of the theorems of the Integral Calculus. It is most important at present, however, not to multiply examples, but to show as soon as possible whence the importance of the point arises.

To find the Motion of the Centre of Mass when the Motions of the Separate Particles are known. Let the distances of the particles from a given plane at any instant be y_1, y_2, y_3, etc. Then the distance of their centre of mass from the same plane is Y; where

$$Y = \frac{m_1y_1 + m_2y_2 + m_3y_3 +}{m_1 + m_2 + m_3 +}. \qquad (1)$$

After an interval, t, let their new positions be represented by accented letters; then

$$Y' = \frac{m_1y_1' + m_2y_2' +}{m_1 + m_2 +}. \qquad (2)$$

Subtract (1) from (2) and divide by t

$$\frac{Y' - Y}{t} = \frac{m_1\frac{(y_1' - y_1)}{t} + m_2\frac{(y_2' - y_2)}{t} +}{m_1 + m_2 +}.$$

The expression on the left is the average velocity (measured perpendicularly to the given plane) of the centre of mass; and $(y_1' - y_1)/t$ is the average velocity in the same direction of the first particle, $(y_2' - y_2)/t$ of the second particle, and so on. Represent these by V, v_1, v_2, etc.; then we have

$$V = \frac{m_1v_1 + m_2v_2 + m_3v_3 +}{m_1 + m_2 + m_3 +},$$

i.e. the average velocity of the centre of mass is obtained by the same formal rule as its position. If the time interval is indefinitely short the average velocities become instantaneous velocities; thus the rule is true for them also.

If the velocities change, let them become values denoted by accented letters in a time interval, t. Subtracting and dividing as before, we get

$$\frac{V' - V}{t} = \frac{m_1 \frac{v_1' - v_1}{t} + m_2 \frac{v_2' - v_2}{t} +}{m_2 + m_2 +},$$

and every term like $(v_1' - v_1)t$ is an acceleration. Denoting these by A, a_1, a_2, etc.

$$A = \frac{m_1 a_1 + m_2 a_2 + m_3 a_3 +}{m_1 + m_2 + m_3 +};$$

or the acceleration of the centre of mass is obtained from the separate accelerations by the same formal rule. Write the total mass, M, then

$$MA = m_1 a_1 + m_2 a_2 + m_3 a_3 +$$

Now $m_1 a_1$ is the component force perpendicular to the given plane acting on the first particle, $m_2 a_2$ that acting in the second, and so on; and MA is a single force which would produce the actual acceleration in the given direction of a particle of mass equal to the total mass of the body and moving with the centre of mass. Now some of the forces acting on the individual particles are internal forces, *i.e.* they arise from the presence of other particles *in the group itself.* These always occur in pairs, for, as we have seen, Action and Reaction are equal and opposite. Thus, if the first particle acts on the second, the second acts on the first with equal force in the opposite sense. When these internal forces are added up they will mutually cancel one another in pairs.

Hence, only the external forces need be considered, *i.e.* those arising from bodies outside the the body under examination. Hence the following important rule :—

To find the acceleration (in any given direction) of the centre of mass of a body, add up the external force-components in the given direction, and divide by the total mass.

If this is done for any two directions making any angle with one another, the actual acceleration of the centre of mass is the resultant of the components so found. It is usually best to choose the directions at right angles to one another.

Cor. If the direction is that of the resultant force, the above rule gives the resultant acceleration.

It is impossible to overestimate the importance of this rule. We see that the motion of a particular point in the body has the same simplicity as that of a particle; thus all the theorems proved with regard to the motion of a particle are applicable. This is true whether the particles are moving as a rigid body or not; for we have imposed no restriction as to their individual motions. However, it is, in general, only in the case of a rigid body that a knowledge of the centre of mass is useful, for in other cases the form of the group is constantly changing, and the position of the centre of mass *within it* is also variable. Thus, if a handful of chaff were thrown upwards in the absence of the air, and so that its centre of mass initially moved vertically, it would continue so to do, for the forces of gravity on the individual parts are all sensibly vertical. The only, but important, difficulty is in specifying the position of the centre of mass among the moving particles of chaff. But in the case of a rigid body it maintains an unalterable position in the body, and once this position is known its motion can be followed. Thus, if a walking stick be substituted for the chaff, and whirled in the air, then under the same initial circumstances we learn that a definite point of the stick will remain on a vertical line, moving up and then downwards just as a small body would do; and this is so whether the whole stick is turning round or not as it moves. [Of course, if the air is present we must include the forces due to it, and these not being necessarily vertical we must not expect the motion to continue so, hence arise the eccentricities of motion of a falling piece of paper.]

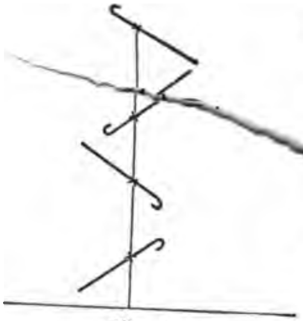


Fig. 53.

The following examples illustrate the above theorems :—

1. A lead ball 10 lb. mass roll
of 10 feet per second in a strai

another of 100 lb. How fast does their centre of mass move?

$$\text{Here}\quad m_1 = 10 \qquad v_1 = 10$$
$$m_2 = 100 \qquad v_2 = 0$$
$$\therefore\ V = \frac{10 \times 10}{10 + 100} = \frac{100}{110} \text{ feet per second.}$$

2. Two stones of masses 4 and 2 are thrown up at the same moment with velocities of 128 and 64 feet per second. Find the initial velocity and the acceleration of their centre of mass; and the time it requires to reach its greatest height.

$$\text{Here initial velocity, } V = \frac{(4 \times 128) + 2 \times 64}{4 + 2}$$
$$= 106\tfrac{2}{3} \text{ feet per second.}$$

$$\text{The acceleration, } A = \frac{m_1a_1 + m_2a_2}{m_1 + m_2} = \frac{m_1g + m_2g}{m_1 + m_2} = g.$$

Thus the centre of mass loses velocity at the rate g; *i.e.* 32 feet per second each second; hence the time required to lose it all is $(106\frac{2}{3})/32 = 3\frac{1}{3}$ seconds. It must be observed that this is not the time of rise of either of the separate masses; the larger one rises for four seconds and the smaller one for only two. Hence the smaller mass has begun to descend before the centre of mass has stopped rising; and the larger one goes on rising though the centre of mass is falling. The greatest height, S, of the centre of mass is given by $V^2 = 2gS$; hence

$$S = \frac{(106\frac{2}{3})^2}{2 \times 32} = 166\tfrac{2}{3} \text{ feet.}$$

The larger mass rises to a height of 256 feet, while the smaller only reaches a height of 64 feet.

3. Two balls, of masses 4 and 6 grams, are acted upon by forces both in the same direction and of values of 100 and 200 dynes; find the acceleration of their centre of mass.

The direction of the acceleration is that of the forces; its value is (100 + 200)/(4 + 6)

$$= \frac{300}{10} = 30 \frac{\text{cms.}}{\text{sec.}^2}$$

4. The forces at right angles to one another, but otherwise the same as (3).

In the direction of the force of 100 dynes the component acceleration is

$$\frac{100}{4+6} = 10 \frac{\text{cms.}}{\text{sec.}^2}$$

In the direction of the force of 200 dynes the component acceleration is

$$\frac{200}{4+6} = 20 \frac{\text{cms.}}{\text{sec.}^2}$$

These two components are at right angles; hence the resultant acceleration of the centre of mass is

$$\sqrt{10^2 + 20^2} = 10\sqrt{5}.$$

EXAMPLES TO CHAPTER IX.

1. Find the centre of mass of four particles whose masses are 4, 3, 2, 1, and which are placed

in order so as to divide a straight line of 24 cms. long into 5 equal parts.

2. Find the centre of mass of three particles of masses 1, 2, 3, placed at the corners of an equilateral triangle.

3. Masses of 1 lb., 8 oz., and 4 oz. are placed at the corners A, B, C respectively, of a right-angled triangle with right angle at B. If the sides $\overline{AB}$ and $\overline{BC}$ are 10 and 12 in., find the distances of the centre of mass from the points A and B.

4. A piece of uniform paper in the form of an equilateral triangle is folded so that one of its angles is at the middle of the opposite side. Find its centre of mass.

5. A uniform lamina is in the form of an equilateral triangle, with one of its sides common to an adjoining square. Find the centre of mass of the combined lamina.

6. If the triangle in (5) is an isosceles one, what must be its height in order that the centre of mass of the combined laminæ may be on the common side?

7. Find the centre of mass of a lamina in the form of half of a regular hexagon.

8. Two bodies, of masses 8 and 16, are thrown up simultaneously from the level of the ground with velocities 128 and 32 ft. per sec. respectively. Find the initial velocity of the centre of mass, the height to which it ascends, and the time of ascent.

9. Show that in an Atwood's machine, when the masses are m_1 and m_2 and are free to move, the velocity of their centre of mass is always downward. Calculate its acceleration and its velocity after t seconds.

10. Thence calculate the external force which must be acting on the two masses, and deduce the value of the tension in the cord.

11. A mass, m_1, lying on a smooth table has attached to it a horizontal string, which passes over a pulley, and then hangs with a mass, m_2, attached at its free end. Calculate in direction and magnitude the acceleration of their centre of mass when they are moving unopposed.

12. Show that the velocity of the centre of mass of two bodies is unaltered by any mutual action between them.

13. How could you get off if placed on the centre of a perfectly smooth, horizontal, dining-room table?

14. Four equal parallel forces, each = 12 dynes, act one on each of four bodies of masses 1, 2, 3, 4 grams respectively. Find the acceleration of their centre of mass.

15. If originally they are at the corners of a square whose side is 10 cms., where will they be after the lapse of 2 sec., and where will their centre of mass be?

CHAPTER X

TORQUE, RESULTANTS, AND COUPLES

WHEN the motion of a rigid body is restricted to a particular plane, any possible case can be represented by the simultaneous occurrence of a translation and a rotation.

For let AB be a line in the body; it can be moved from the position AB to A′B′ in two steps, the first consisting of a translation parallel to itself till A reaches A′; at the same moment B reaches *b*, where B*b* = AA′.

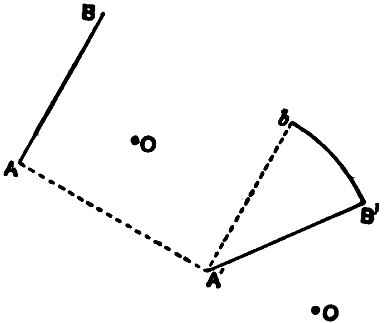


Fig. 54.

The line can now be turned round A′ as centre till *b* reaches B′. Or AB may be rotated round A till it is parallel to its final position, and then moved parallel to itself till A reaches A′, at which moment B will reach B′. Or the rotation may be about any other point O as centre, followed or preceded by a corresponding translation. Or again, since the two motions are independent of one another, they can take place simultaneously with the same final result. It should be observed that in each of the above cases the total rotation is the same, but the magnitude of the translation depends upon the particular point O selected. Now we have seen how to find the translatory motion of the centre of mass; we completely solve the case of motion in one plane if we can find the rotation taking place round it. *This can be deduced from the laws of motion already given.* But, in an introductory treatise, it is better to start with fresh experimental facts, on which to base the theorems of rotation. Even after the general deduction is given, these experiments can be taken as further verification of the theorems already given, with which they are in accordance. This is the method which we shall adopt.

i. The *torque*, or power of a force in producing rotation, depends not only on its magnitude, but also on its *position.* Thus, let a string be attached in succession to various points of a block resting on a smooth table; when it is pulled, opposite rotations or translation without rotation is produced, according to the point of attachment, although the direction of pull be always the same. It is not hard to verify that the case of no rotation occurs when the direction of the string passes, at any rate approximately, through the centre of mass.

ii. A graduated bar is balanced on a knife edge, so placed that the weight of the bar produces no rotation. We can then examine the effect of placing an additional weight anywhere on the bar. Placed on the side AO it produces counter-clockwise rotation, on the side OB clockwise rotation.

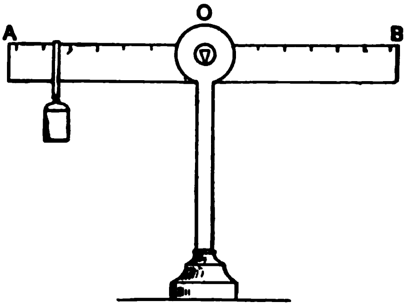


Fig. 55.

iii. Two equal weights placed at equal distances from O on opposite sides annul each other's effects whatever the distance may be.

iv. Unequal weights are equivalent when their distances are inversely as the weights.

v. Any number of weights on one side can be balanced by a single weight on the other. If the product of each weight into its distance is obtained, it is found that the product for the single weight is the sum of the other products.

vi. In the above cases the force is perpendicular to the bar; if it is not so, it is the perpendicular component alone which is effective; or, what comes to the same thing, it is the force into its *perpendicular distance* which is effective. Hence, we conclude that, at any rate, for the case of a fixed axis, the turning power of a force is proportional to the perpendicular distance of its line of action from the axis.

The product of the force into this perpendicular distance is called its *moment.* Thus, if O be the axis and F the force, as shown in Fig. 56, the moment is Fd.

If the diagram of positions is full size, and F and d are on the same scale (*i.e.* unit length representing unit force), the moment is equal to twice the area of the triangle OAB. In any other case this area must be multiplied by twice *the physical equivalent of unit area.* This equivalent is equal to kl where unit length represents k units of force, and l units of distance.

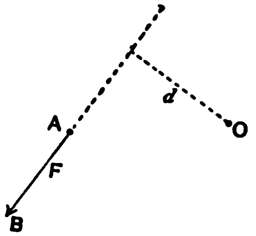


Fig. 56.

It is not so easy to make experiments when the axis itself is moving in space (though fixed in the body); as, for example, when a wheel rolls on the ground. But observation, as well as the laws of motion of particles, show that the turning power is then proportional to its moment *only when the axis passes through the centre of mass.* This latter, however, is the most important case, for we have found the translation of the centre of mass; it is the rotation round it which we require to find in order to complete the determination of its motion.

We may now make several deductions :—

Any force is equivalent to another force

(1) If each produces the same acceleration of the centre of mass.

(2) If each produces the same rotation round it.

To satisfy the first, their magnitudes and directions must be the same. To satisfy both, their lines of action must also be the same. Unless *both* are satisfied, the given forces are not equivalent.

It follows that a force may be considered as acting at any point of its line of action produced. For if moved along this line, its magnitude, direction, and moment all remain unchanged. This fact is sometimes known as the transmissibility of force.

If two forces in a plane are inclined to one another, their lines of action, produced if necessary, intersect in a point. The forces may be considered as acting at this point.

Any force is equivalent to a given set of forces in one plane if its magnitude and direction are those of their resultant (to satisfy i.), and if its moment is equal to the sum of their moments with respect to a given axis (to satisfy ii.).

If two forces act at a point, the single force equivalent to them also acts through the same point. To prove this we require to show that a force equal to their vector-sum has then a moment equal to the sum of their moments about any axis.

Let O be the point of intersection, and let P represent the axis about which moments are to be taken. Draw the forces on such a scale that $\overline{AC}$ parallel to $\overline{OB}$ passes through the point P.

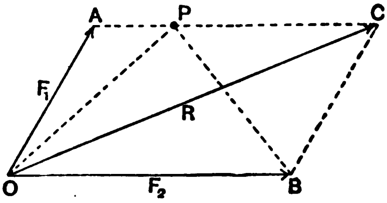


Fig. 57.

The moment of F_1 is $2\kappa \times$ area OAP, where κ is the physical equivalent of unit area; the moment of F_2 is $2\kappa \times$ area OBP, and the moment of R (their resultant) is $2\kappa \times$ area OCP; areas being considered of opposite sign when oppositely gone round.

$$\text{Now area OBP} = \text{area OBC} = \text{area OCA}$$
$$= \text{area OCP} + \text{area OPA}.$$
$$\therefore \text{area OCP} = \text{area OBP} + \text{area OAP}.$$
$$\text{And} \therefore \text{moment of R} = \text{moment of } F_2 + \text{moment of } F_1.$$—Q. E. D.

If the point P lies outside the angle between F_1 and F_2 the same result holds, as the student should show; thus the theorem is always true.

Hence, if two forces in a plane are inclined to one another, shift them if necessary along their lines of action till they intersect. Find their resultant by the triangle law. If the resultant be placed so that its line of action produced passes through the same point of intersection, it will be equivalent to the two forces *in every respect.*

Note: If it be placed in any other position it will produce the same acceleration of the centre of mass (for this only depends upon the magnitude and direction); but the rotation produced round the centre of mass will be different.

If there are more than two forces inclined to one another, two can be replaced by one as above, and then this and the third replaced by one by the same method, and so on.

In carrying out this method it is best to draw two figures, one of which is the ordinary polygon of forces which determines the direction and magnitude only; the other diagram is for fixing the position of the resultant. Thus for three forces F_1, F_2, F_3, acting at A, B, and C, proceed as follows:—

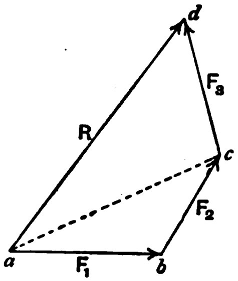


Fig. 58 (i).

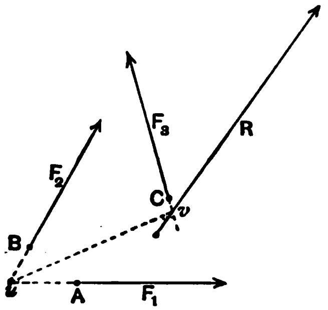


Fig. 58 (ii).

Construct the polygon of forces *abcd*; then *ad* represents the resultant, except in regard to position. In Fig. 58 (ii) produce F_1 and F_2 to their point of intersection *u*; through *u* draw a line parallel to *ac*, cutting F_3 in *v*; through *v* draw a line parallel to and *equal to ad*; then this is the resultant required. The direct application of this method fails when the forces are parallel for the points of intersection are then at an infinite distance.

A more general method (of which this is a special case) is commonly employed in practice, and is applicable whether the forces are parallel or not.

Instead of drawing lines parallel to lines starting from *a*, any arbitrary starting point, O,

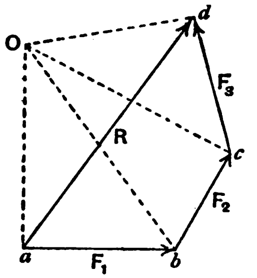


Fig. 59 (i).

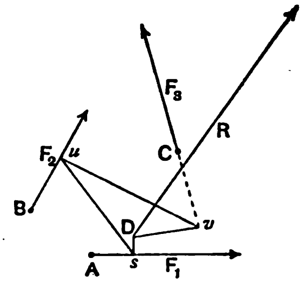


Fig. 59 (ii).

may be employed which is not on the line *ad* unless at either end, Fig. 59 (i) and (ii).

From any point *s* in F_1 (Fig. 59 ii) draw a line parallel to $\overline{Ob}$, cutting F_2 in *u*; from *u* draw a line parallel to $\overline{Oc}$ cutting F_3 (produced if necessary) in *v*; from *v* draw a line parallel to $\overline{Od}$, and from *s* draw a line parallel to $\overline{Oa}$; the point of intersection D of the last two lines is a point on the line of action of the resultant; a line drawn through it parallel and equal to $\overline{ad}$ will therefore represent the resultant completely. The polygon *suv*D (excluding the closing side, $\overline{sD}$) is called the funicular or link polygon of the set of forces.

To prove this method it must be noted that the force F_1 can be resolved into two components $\overline{aO}$, $\overline{Ob}$, whose lines of action are $\overline{sD}$ and $\overline{us}$ respectively; similarly F_2 can be resolved into $\overline{bO}$, $\overline{Oc}$, whose lines of action are *su* and *uv*; and it must be observed that $\overline{Ob}$ and $\overline{bO}$ neutralise one another. Similarly F_3 is resolvable into $\overline{cO}$ and $\overline{Od}$ whose lines of action are $\overline{vu}$ and $\overline{Dv}$. Since $\overline{Oc}$ and $\overline{cO}$ also neutralise one another the three forces are equivalent to two whose lines of action are $\overline{sD}$ and $\overline{Dv}$; therefore the resultant of the three forces must pass through D the point of intersection of these two.

A case of failure of these methods will be considered shortly (p. 159).

To solve the problem by calculation, the magnitude and direction of the resultant are determined as for forces meeting at one point (p. 79); the position is then chosen so that the moment of the resultant is equal to the sum of the moments of the several forces about any one point. If the perpendicular distances d_1, d_2, etc., of these forces from the point are known, the moments are F_1d_1, F_2d_2, etc., which must be added together—with proper sign—and divided by R, the resultant; the quotient is the distance, D of the resultant from the given point.

If the rectangular co-ordinates of any point on the line of action of any force is given, the distance of this force from the origin is $d = (y \cos \theta - x \sin \theta)$, where θ is the inclination of the force to the axis of x; this expression multiplied by F gives the corresponding moment.

PARALLEL FORCES.

The important case for which the forces are parallel will now be dealt with in greater detail.

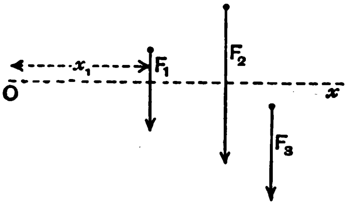


Fig. 60.

1. If several forces F_1, F_2, F_3, etc., are parallel their resultant is a parallel force, R, equal to their algebraic sum. Let Ox be any line perpendicular to the forces, and x_1, x_2, x_3, etc., the distances of the forces respectively from O; then the distance, X, of the resultant is

$$X = \frac{F_1x_1 + F_2x_2 + F_3x_3 +}{F_1 + F_2 + F_3 +}$$

If some of the forces are in opposite directions to others they must be taken of opposite sign; and again, if they lie on opposite sides of O the values of x must also be taken of opposite sign.

Special case: Two forces *numerically* equal to F_1 and F_2 at a distance apart $= d$.

By taking O on F_1 itself we get

$$R = F_1 + F_2. \qquad X = \frac{F_2 d}{F_1 + F_2}$$

if the forces are of the same sign; *i.e.* the resultant is a single force $F_1 + F_2$ which divides the distance between the forces into parts inversely as the forces adjacent to them.

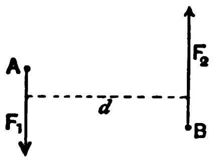


Fig. 61.

If they are of opposite sign, as shown (Fig. 61), and F_2 is the greater, the resultant is $F_2 - F_1$ in the direction of F_2, and its counter-clockwise moment about O is F_2d.

$$\therefore X = \frac{F_2d}{F_2 - F_1}$$

Since $F_2 - F_1$ is necessarily less than F_2, X is always greater than d, *i.e.*, the resultant lies outside the given forces on the side of the greater force. As F_2 approaches F_1 in magnitude, the resultant tends towards a zero value, but at the same time X becomes exceedingly large; it would be erroneous to conclude, therefore, that the effect of the force vanishes. Their total moment about any point O from which they are distant x_2 and x_2 respectively is $F_2x_2 - F_1x_1 = F(x_2 - x_1) = Fd$, when they have actually become equal. In this case it is impossible to replace the forces by a single finite resultant; the two forces are then dealt with together, and are called a *couple*.

Definition.—*Two equal opposite forces whose lines of action are not the same are called a couple.*

Properties of a Couple. Since in the case of a couple the resultant is zero, it can produce no acceleration of the centre of mass of the body on which it acts.

Its moment about any point is equal to the product (F*d*) of one of the forces into the perpendicular distance between them, and is therefore the same about any point in their plane. This perpendicular distance is called the ***arm*** of the couple.

Since a couple, if it exists, has a finite moment, it will produce rotational acceleration round the centre of mass.

The moment of a couple is its important characteristic, because it is a rotator simply; and indeed all couples in one plane are equivalent to one another provided their moments are equal, for it is only their rotation-reducing power which has to be considered. The following cases may be specially stated in order to illustrate the importance of this proposition.

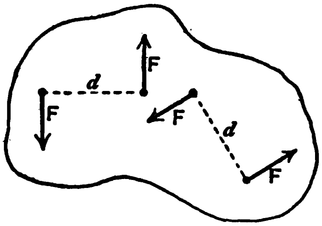


Fig. 62.

1. A couple may be supposed moved to any part of its plane.

Thus, the two couples shown in Fig. 62

produce exactly the same effect upon the body on which they act.

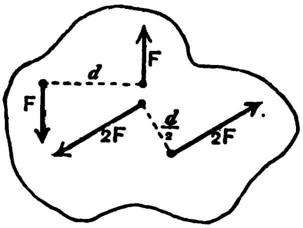


Fig. 63.

2. A couple may be replaced by another of equal moment acting anywhere in its plane.

Thus, the two couples shown are exactly equivalent because $Fd = 2F \times \frac{d}{2}$.

3. The resultant of two couples in the same plane is another couple in the same plane.

For the resultant of all four forces is obviously zero, since they cancel each other's magnitudes in pairs; hence they will be equivalent to a couple of moment equal to the sum of their moments unless this sum is zero, in which case they annul each other completely.

4. A single force acting at any point of a rigid body may be replaced by an equal force acting at any other point of the body together with a couple.

For the resultant force is the same in both cases; and the sum of the moments about any point can be made the same by suitably choosing the couple. Thus F at B can replace F at A, if a clockwise couple be introduced whose moment is Fd.

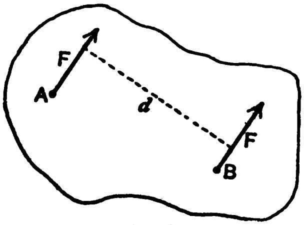


Fig. 64.

It will be seen that we have based the proof of these theorems on the theorems of resultants and moments.

It is more usual in elementary statics to give proofs which only involve these principles tacitly. Thus (1) may be proved as follows :—

Let the equal couples be $\overline{PP'}$, $\overline{QQ'}$ respectively (Fig. 65). Now, if they are equivalent, one of them reversed will cancel the other; let $\overline{QQ'}$ be reversed. Produce the lines of action of all four forces, so as to form the equilateral parallelogram *abcd*. The resultant of P and minus Q will pass through *b*, and since P and Q are equal its line of action will bisect the angle *abc*, and is therefore along *db* from *d* to *b*. Again, the resultant of P′ and minus Q′ will pass through *d*, and because P′ and Q′ are equal it will bisect the angle *adc*, that is, it is along *bd* from *b* toward *d*. The resultants being equal in the two cases, and having the same line of action, but being oppositely directed along it, will thus neutralise each other completely.

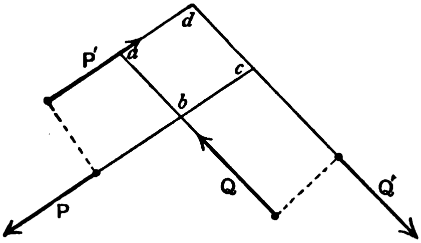


Fig. 65.

In this proof the principle of moments enters tacitly in the assumption that the line of action of a resultant of two forces acting on a rigid body passes through the point of intersection of its components produced.

Again (4) is proved as follows:—Let F be moved along its own line so as to act at A′, a point on the perpendicular to itself through B. Introduce two equal opposite forces F and F′ at B′ each equal and parallel to F; these will not have any effect on the motion of the body. But F at A′ and F′ at B constitute a couple, and there remains the force F at B; hence the theorem is proved. The principle assumed here is the principle of transmissibility which we have based upon the principle of equivalence of moments.

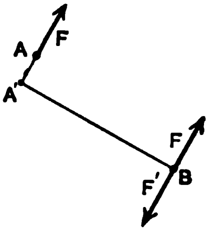


Fig. 66.

The methods given for finding the resultant of a given set of forces (p. 151) will, of course, fail when this simplest equivalent is a couple; for the two final lines *v*D and *s*D on the funicular are then parallel to one another—*i.e.* their point of intersection is at infinity. But, in this case, the couple is equivalent to a force, *a*O, acting at *s*, and an equal force, O*a*, acting at *v*. If therefore there is no couple as well as no resultant—in which case the set of forces has no dynamic effect—the point D must correspond with the point *s*; that is the funicular is closed as well as the polygon of forces. And conversely, if both of them are closed the set of forces has no dynamic action. Such a set of forces is said to be *in equilibrium*; or frequently this term is also applied to the body upon which they act. We shall deal with this case in extenso in Chapter XI.

The gist of these theorems can be summarised in a single statement: Every set of forces acting in one plane can be reduced either to a single force acting through a determinate point, or else to a single couple acting anywhere in their plane, provided that the set of forces is not without dynamic action, in which case both force and couple are zero.

Centre of Parallel Forces. In several cases the forces acting on a body are localised in such a way that when the body moves they move with it. Thus the gravity of any particle must be, as it were, attached to the special particle; so that, if we were to sketch in the gravitational forces on an extended body, these would be represented by a group of vertical lines acting at points fixed in the body. If the body turns round, the forces still act on the same particles, but *remain vertical.* The following theorem is true for all such groups of parallel forces:

If a set of parallel forces act at points fixed in a body, and be turned through any angle—the same for all—with respect to the body, their line of action always passes through a certain point called their *centre.*

Here the forces are supposed to turn and the body to remain fixed; it is of no consequence how this *relative* motion takes place. The theorem is proved thus :

Let there be two parallel forces F_1, F_2 acting at A and B (Fig. 67); their resultant is a force, $F_1 + F_2$, whose line of action divides the line AB into parts such that $\frac{AC}{CB} = \frac{F_2}{F_1}$.

Let now the forces be turned through any angle—the same for both—about the points A and B; their resultant again divides the line AB at the same point, C. Since this is true whatever the angle is, the point C is a fixed point, through which the resultant passes; this is the *centre* of the two forces.

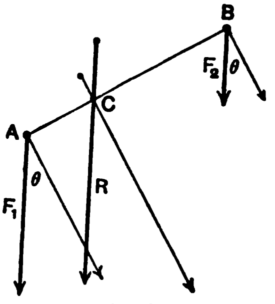


Fig. 67.

It is plain that, if there is a third force acting at D, then the three have a centre; for F_1 and F_2 have a resultant which always passes through C; thus we have to deal with parallel forces acting at fixed points, C and D, to which the same theorem applies, and so on for any number.

A methodical way of finding the centre is to find the line of action of the resultant of the forces in any two angular positions; the point of intersection of these resultants must be the centre required. It is usually most convenient to take the positions at right angles to one another. Thus, let the parallel forces act at A, B, C, etc. Take any two lines O*x* and O*y*.

Imagine the forces to be all parallel to O*y*. Then the perpendicular distance of their resultant from O*y* is

$$X = \frac{\Sigma(\text{moments with respect to a point in } Oy)}{\Sigma(\text{forces})}.$$

Now, imagine them all acting parallel to O*x*; the perpendicular distance of their resultant from O*x* is

$$Y = \frac{\Sigma(\text{moments with respect to a point in } Ox)}{\Sigma(\text{forces})}.$$

X and Y are the co-ordinates of the *centre*. It should be noticed that the moments referred to are also equal in each case to the product of a force into the perpendicular distance of its *point of action* from a *given line*, such as O*x* or O*y*.

Centre of Gravity. As long as a body is of moderate dimensions, the forces due to gravity acting on its separate parts may be considered as a group of such localised parallel forces. The centre of force is in this case called the centre of gravity. The distance of its position from any fixed line given by the above rule is

$$X = \frac{\Sigma(\text{product of a weight into perpendicular distance from } Oy)}{\Sigma(\text{weights})}$$
$$= \frac{\Sigma(w\,.\,x)}{\Sigma w},$$

where *x* stands for the perpendicular distance of any particular weight, *w*, Now $w = mg$ where m is the mass of the particle whose weight is w. Hence

$$X = \frac{\Sigma(mgx)}{\Sigma(mg} = \frac{\Sigma(mx)}{\Sigma(m)},$$

since g is a constant.

Similarly Y may be written $\frac{\Sigma(my)}{\Sigma(m)}$.

Now these are also the equations for determining the centre of mass of the body. Hence the centre of gravity and centre of mass of a body coincide. For this reason the centre of mass is often known as the centre of gravity. It will tend to prevent confusion of ideas if the names are kept quite distinct, even though the same point is denoted by each.

It should also be borne in mind that the weights of different parts of a large body are not parallel; thus, strictly speaking, the idea of a centre of *gravity* only represents an approximation sufficiently true for small bodies. This restriction does not apply to the centre of *mass*.

The importance of knowing the position of the centre of gravity arises from the possibility of replacing the forces due to gravity by a single force equal to the total weight, whose line of action always passes through the centre in whatever position the body may be placed.

We conclude this chapter with two worked out problems.

(i) Three forces are represented completely by the sides of a triangle taken the same way round: show that they are equivalent to a couple, and find its moment.

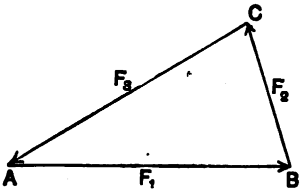


Fig. 68.

Let ABC be the triangle. Since it is a closed figure the resultant force is zero; hence the given forces are equivalent to a couple, unless their moment sum vanishes about any point. Take moments about A; F_1 and F_3 have each no moment, and F_2 has a moment represented by twice the area of the triangle; thus this is also the moment of the couple.

Or thus: The resultant of F_1 and F_3 acting at A is a single force equal and opposite to F_2 acting at A; thus the three are equivalent to two equal opposite forces, one at A and the other at B—*i.e.* to a couple.

(ii) Three forces are represented *completely* by three sides of a rectangle taken the same way round; find their resultant.

Let the rectangle and forces be as shown (Fig. 69).

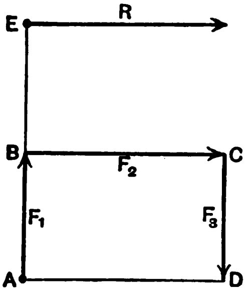


Fig. 69.

The resultant is obviously a single force represented in magnitude and direction by the side AD—*i.e.* it is equal to F_2. Take moments about A; the sum is $F_2 \times AB + F_3 \times AD$; divide this by the resultant and its distance from A is obtained. Now, since BC represents F_2 and CD represents F_3, the sum of the moments is twice the area of the rectangle; divide this by AD, and the quotient is twice AB; thus the position of the resultant R is twice AB, reckoned from A as shown.

The student must particularly observe that the closing side of the polygon does not generally represent the resultant forces *in position.*

EXAMPLES.

1. If forces are represented *completely* (*i.e.* in magnitude, direction, and position) by the sides of a triangle taken the same way round, show that they are equivalent to a couple, and calculate its moment.

2. Show that this is also true if the forces are represented by the sides of any rectilinear figure.

3. Three forces are completely represented by the three sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ of a rectangle ABCD. Find a single force that is completely equivalent to them.

4. Show that they are equivalent to a single force completely represented by the side $\overline{AD}$, together with a couple of moment equal to twice this force into the length AB.

5. Solve 3 and 4 for the case in which $\overline{AB}$ is 3 ft. and $\overline{BC}$ is 4 ft., while the forces (in lb. wt.)

which they represent are ten times their length in ft.

6. Parallel forces of 5 and 12 units act 119 in. apart; find the position of their resultant (i) if their directions are the same, (ii) if they are opposite.

7. Parallel forces of 2, 3, and 4 lb. wt. respectively act at the corners of an equilateral triangle whose altitude is 10 in.; find their centre.

8. Parallel forces of 3, 7, and 9 lb. wt., all of the same sign, act so as to divide a straight line 24 cms. long into 4 equal parts. Find where a single force of 38 lb. wt. must be placed in order that the resultant of all may pass through the middle of the line.

9. Three equal forces are parallel to the sides of an equilateral triangle, but act at the beginning, middle, and end of a straight line 120 cms. long, which is parallel to the force acting at the beginning. Show that the forces are equivalent to a couple, and find its moment.

10. In question 9, if the forces are of magnitude 12, 18, and 24, but other data are the same, what are the forces equivalent to?

CHAPTER XI

EQUILIBRIUM OF A RIGID BODY

THE principles which have been developed will now be applied to those particular cases in which there is equilibrium. It must be remembered that we are only considering forces acting in one plane. The conditions that must be satisfied for equilibrium are these two:—

1. The total component of the forces in each of any two directions must vanish.
2. The sum of the moments of the forces about any *one* point in their plane must also vanish.

The satisfaction of the first condition ensures that the resultant force shall be zero, and hence that the centre of mass shall have zero acceleration. If the second is satisfied as well as the first there is also zero rotational acceleration.

If the second condition alone is satisfied, it does not follow that there will be no rotational acceleration. This would only be the case if the point with respect to which moments are taken is the centre of mass. To illustrate this take the case of a single force, F, acting at a point A (Fig. 70). Its moment about a point, P, on its line of action is zero, and yet such a force will produce rotation round the centre of mass. If the first condition is satisfied the single resultant, F, must be zero; the forces are then equivalent to a couple or to nothing at all. Now the moment of a couple is the same about any point in its plane; hence, if the moments vanish about any one point when the first condition is satisfied, they must also vanish about the centre of mass, and there will be no rotational acceleration.

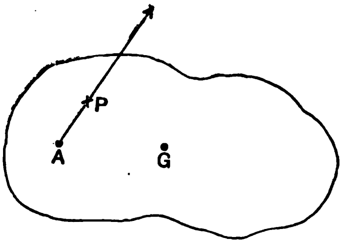


Fig. 70.

If the second condition were altered so that the words "any one point" were replaced by "the centre of mass" the satisfaction of this condition *by itself* would involve the absence of rotational acceleration.

These two conditions may be replaced by a single condition which involves both of them:

Alternative Condition of Equilibrium.—The sum of the moments of the forces about *each* of any THREE points in their plane and not in the same straight line must be zero.

For suppose that there is a single resultant, F; any *two* points may chance to be on the line of action of F; the moments would each be zero, although F is not zero. But in this case the third point must be off the line of action, otherwise it would be on the same straight line as the two others. The moment can only vanish about it also if F itself is zero; thus the first condition is satisfied. The second condition is explicitly satisfied.

The foregoing conditions are quite general: in certain particular cases they can be differently stated:

(*a*) If there are only two forces acting on a rigid body in equilibrium, they are equal and opposite (to satisfy 1) and have the same line of action (to satisfy 2).

(*b*) If there are only three forces, and they are inclined to one another, they must be representable in magnitude and direction by the sides of a triangle taken the same way round (to satisfy 1); and their lines of action must pass through one point (to satisfy 2). If they are parallel to one another each must be equal and of opposite sign to the algebraic sum of the other two; and condition 2 must be explicitly satisfied.

(*c*) If there are more than three, any one must be equal and opposite to the resultant of all the others, and have the same line of action.

Four or more forces in equilibrium *need not*, but *may*, pass through one point. If all but one do, the remaining one *must*.

EXAMPLES IN EQUILIBRIUM.

Application will now be made of the various theorems relating to the equilibrium of a rigid body. The student, in attempting to solve any problem, should always start by sketching on a diagram the forces that are present.

1. A beam resting on two piers, supposed smooth.

The forces are: the weight, W, of the beam acting through the centre of gravity, vertical forces, R_1; and R_2 due to the supports; these will be vertical, since the supports are smooth.

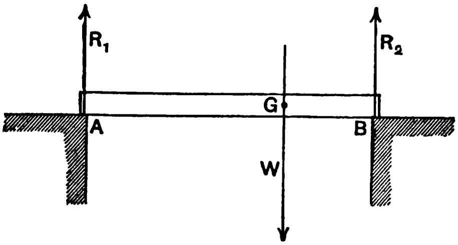


Fig. 71.

Since these forces are parallel

$$R_1 + R_2 - W = 0$$

to satisfy the first condition.

Again, taking moments about the point A,

$$R_2 \times \overline{AB} - W \times \overline{AG} = 0$$

to satisfy the second condition.

Here we have two equations from which to find the two unknown quantities, R_1 and R_2; and these equations are therefore sufficient. The values given by these are

$$R_2 = \frac{\overline{AG}}{\overline{AB}} W \text{ and } R_1 = \frac{\overline{GB}}{\overline{AB}} W.$$

The same result is obtained without explicitly satisfying the first condition if moments are taken about each of the points A and B, and in each case equated to zero. The student should consider why two points are here sufficient instead of the three required in general.

2. A thin uniform beam of length *2l* resting on a smooth peg against a smooth wall. The action R_1 at the wall is perpendicular to it, and the action at the peg is perpendicular to the beam, both being smooth. These forces intersect at a point. The only other force is the weight, W, of the beam, which must therefore act though the same point, *i.e.* the centre of gravity must be vertically above it. Consider the triangle ACB; its sides are parallel to the three forces, they must therefore be proportional to them. Let θ be the inclination of the beam to the horizontal; a the distance of peg from wall.

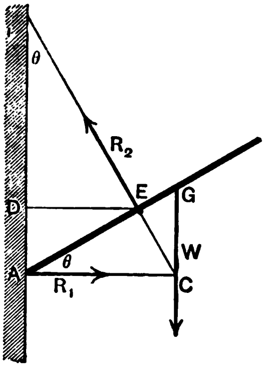


Fig. 72.

The three triangles ADE, CEA, and GCA are similar; therefore $\frac{\text{DE}}{\text{AE}} = \frac{\text{AE}}{\text{AC}} = \frac{\text{AC}}{\text{AG}}$,

$$\therefore \left(\frac{\text{DE}}{\text{AE}}\right)^3 = \frac{\text{DE}}{\text{AE}} \cdot \frac{\text{AE}}{\text{AC}} \cdot \frac{\text{AC}}{\text{AG}} = \frac{\text{DE}}{\text{AG}} = \frac{a}{l}.$$

$$\therefore \cos\theta = \frac{\text{DE}}{\text{AE}} = \sqrt[3]{\frac{a}{l}},$$

which determines the position of equilibrium. The magnitudes of the forces are given by

$$\frac{\text{R}_1}{\text{W}} = \frac{\text{AC}}{\text{AB}} = \tan\theta; \quad \frac{\text{R}_2}{\text{W}} = \frac{\text{BC}}{\text{AB}} = \frac{1}{\cos\theta}.$$

Alternative method. Resolve the forces horizontally—

$$\text{R}_1 - \text{R}_2 \sin\theta = 0. \qquad (1)$$

Resolve vertically—

$$\text{R}_2 \cos\theta - \text{W} = 0. \qquad (2)$$

Take moments about A—

$$\text{R}_2 \times \overline{\text{AE}} = \text{W} \times \overline{\text{AC}}. \qquad (3)$$

But $\overline{\text{AE}} = \frac{a}{\cos\theta}$ and $\overline{\text{AC}} = \overline{\text{AG}} \cos\theta = l \cos\theta$.

$\therefore$ (3) becomes $\text{R}_2 \times a = \text{W}\, l \cos^2\theta$.

Combine this with (2):

$$\frac{a}{l} = \cos^3\theta.$$

If equilibrium does not exist, the forces will not meet at a point, but their line of action will enclose a triangle such as C_1, C_2, C_3 (Fig. 73). The student must not hastily conclude that the moment tending to turn the beam into or away from the position of equilibrium is given by twice the area of this triangle; for, although its sides are parallel to the three forces, we cannot say that they are proportional to them *because equilibrium does not exist.*

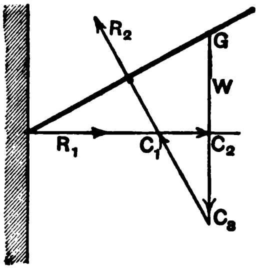


Fig. 73.

3. Uniform horizontal beam (length $2l$) inserted at one end in a wall. To simplify the conditions, let the shape of the opening be as shown, in order that the action of the wall may be localised as much as possible. The forces are the weight at G, and forces due to the wall at B and A. Suppose the quite possible case that these are vertical, then the force at A must be downwards (to balance the moment of W about B), and that at B must be upwards, and equal to the sum of the other two—*i.e.*

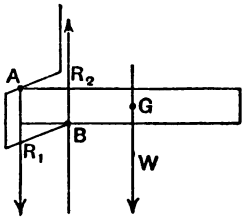


Fig. 74.

$$R_2 = R_1 + W;$$

and taking moments about A,

$R_2 a = Wl$, where a is the horizontal distance between A and B;

$$\text{whence} \quad R_2 = \frac{Wl}{a} \text{ and } R_1 = W\frac{l-a}{a}.$$

If it be desired to replace R_2 and R_1 by a single force, it must obviously be equal but opposite to W, and have the same line of action. Thus the resultant of the forces due to a wall upon the beam imbedded in it *does not act through the wall, but at a point outside it.* This arises from some of the component forces being oppositely directed to others. We have seen that when there are two oppositely directed forces, their resultant lies outside them and on the side of the greater of them (p. 155); and this may equally well take place when there are more than two. Some caution has to be exercised then in placing the resultant force due to a support which acts over a finite area. The components of course act at the wall; but certainty that their resultant does so exists only when it is obvious that all the components have roughly the same direction.

In all cases the resultant may be placed at the wall *if a suitable couple is also introduced* (p. 157); thus, if the resultant $R_2 - R_1$ is placed at A, the couple must have a counter-clockwise moment, R_2a.

It must also be borne in mind that R_1 and R_2 *may* not be vertical, though their resultant must be so in order to counteract W. Hence their horizontal components, if there are any, must be equal and opposite.

4. Ladder resting against a smooth wall and on a smooth ground. Find the horizontal force that

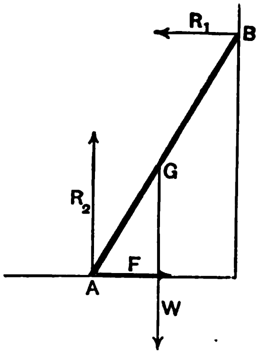


Fig. 75.

must be applied at its lower end to keep it from slipping. The forces are as shown (Fig. 75) where F is the force required.

Resolve vertically

$$R_2 = W.$$

Resolve horizontally

$$F = R_1.$$

Take moments about A

$$R_1 \times \overline{AB} \sin \theta - W \times \overline{AG} \cos \theta = 0.$$

$$\therefore R_1 = W\frac{\overline{AG}}{\overline{AB}} \cot \theta,$$

and this also equals F.

5. A rod resting on two smooth inclined planes; find the actions at the ends (Fig. 76). R_1 and R_2 are normal to the two planes. Draw a horizontal line $\overline{CD}$; then the sides of the triangle CDN are perpendicular to the three forces; if it be turned

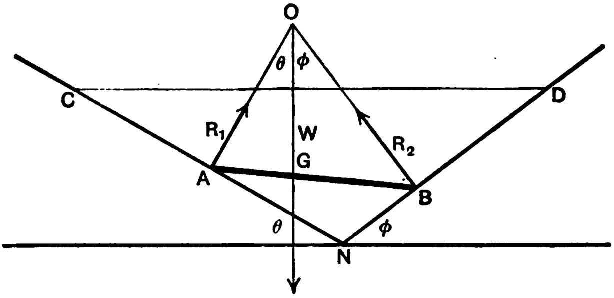


Fig. 76.

through a right angle they will be parallel, and therefore proportional to them—*i.e.*

$$\frac{\overline{CN}}{R_1} = \frac{\overline{DN}}{R_2} = \frac{\overline{CD}}{W}.$$

Now the angle at C is θ, and at D is ϕ, where $\theta \times \phi$ are the angles of the planes.

$$\therefore R_1 = W\frac{\overline{CN}}{\overline{CD}} = W\frac{\sin\phi}{\sin(180^\circ - \theta - \phi)} = W\frac{\sin\phi}{\sin(\theta + \phi)}.$$

and similarly $R_2 = W\frac{\overline{DN}}{\overline{CD}} = W\frac{\sin\theta}{\sin(\theta+\phi)}$.

These forces are determined without making use of the equation of moments, because their directions do not depend upon the position of the rod. This position is determined by making the sum of the moments vanish. Since to satisfy this condition the forces must intersect in one point, the centre of gravity of the rod must be vertically below the point of intersection of R_1 and R_2.

Equilibrium of Frameworks.—When several beams are jointed together they constitute a framework. If the whole framework is at rest each part of it is so also. Consequently the conditions of equilibrium must be satisfied by each part as well as for the whole. This is known as the principle of *separate equilibrium.* By applying these conditions, sufficient equations can be obtained for determining the actions at the joints, provided these are smooth. Such a joint may be represented by a single hinge pin passing through both the beams which it joins; we will suppose the pin cylindrical (Fig. 77).

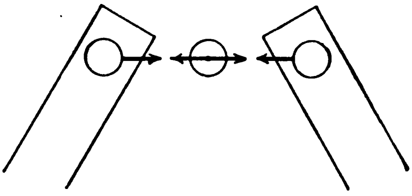


Fig. 77.

Wherever it comes into contact with either beam there is a possibility of a force arising. If it is perfectly smooth all these forces must be normal to its surface; or, in other words, must pass through its centre. Thus the resultant action of either beam upon the pin must be a single force passing through its centre.

If we can neglect the weight of the pin, these resultants for the two beams must be equal and opposite to one another. Since the reactions on the two beams are equal and opposite to their actions on the pin, we can ignore the intervening pin and imagine the beams acting directly upon one another with equal opposite forces whose lines of action pass through the joint, provided that there is no external force acting on the pin other than those due to the beams in contact with it. It is most convenient to resolve these actions into rectangular components.

Example: Two uniform beams, jointed smoothly at A, rest with their ends B and C on a smooth horizontal table. If B and C are connected by a string find the tension in the string, the whole being at rest.

The forces are as shown in Fig. 78.

At A the components on the left beam are equal and opposite to those on the right beam.

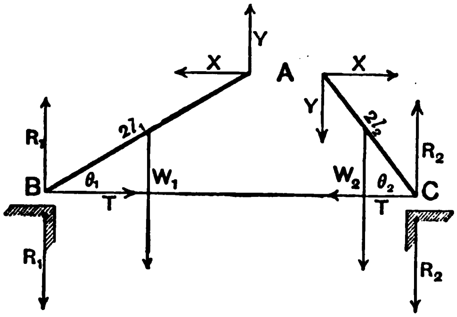


Fig. 78.

W_1 and W_2 are the weights of the beams and T is the tension in the string.

Taking moments about A of the forces on the left-hand beam, we obtain

$$R_1 2l_1 \cos\theta_1 - T 2l_1 \sin\theta_1 - W_1 l_1 \cos\theta_1 = 0,$$

or the equivalent equation,

$$R_1 - T\tan\theta_1 = \frac{W_1}{2}.$$

Similarly for the right-hand beam,

$$R_2 - T\tan\theta_2 = \frac{W_2}{2}.$$

Add these two:

$$R_1 + R_2 - T(\tan\theta_1 + \tan\theta_2) = \frac{W_1 + W_2}{2}.$$

But considering the equilibrium of the whole structure we have, equating vertical components to zero,

$$R_1 + R_2 = W_1 + W_2.$$

Whence

$$T = \frac{W_1 + W_2}{2(\tan\theta_1 + \tan\theta_2)}.$$

If the action at the joint is also required we proceed by determining its two components. Since there must be neither horizontal nor vertical force on left-hand beam,

$$X = T,$$

$$Y = W_1 - R_1 = W_1 - \frac{W_1}{2} - T\tan\theta_1$$

$$= \frac{W_1\tan\theta_2 - W_2\tan\theta_1}{2(\tan\theta_1 + \tan\theta_2)}.$$

The resultant action at the joint is $\sqrt{X^2 + Y^2}$, and the angle ϕ it makes with X is given by

$$\tan\phi = \frac{Y}{X} = \frac{W_1\tan\theta_2 - W_2\tan\theta_1}{W_1 + W_2}.$$

If $\theta_1 = \theta_2$, in which case $W_1 = W_2$, the lengths of the beams being equal, Y is zero; that is, the action at the joint is horizontal. This might be inferred from first principles: for symmetry requires that the action on each beam should be the reverse, with respect to the line of symmetry, of the action on the other, while the law of equality of action and reaction requires that their lines of action shall be the same. Since the line of symmetry is vertical these requirements can both be satisfied only if the action is horizontal.

EXAMPLES.

1. Three equal uniform bars are freely hinged together and suspended by a string attached to the middle point of one of them. Find the action at each joint in terms of the weight of a bar.

2. If the string is attached to the hinge pin at one of the joints, find the action at each joint.

3. Two equal bars are freely hinged together and their free ends are connected by a string of length equal to that of each bar. If the whole be then suspended by a smooth pin through one of the bars close to its free end, find the action at the joint.

4. If the whole be suspended by a string fastened at the centre of gravity of one of the bars, what will be the action at each attachment?

5. Two equal rods AB and BC are smoothly hinged at B, and the end A is then attached at A by a smooth hinge to a vertical wall. At what point of BC must a vertical force be applied to maintain the rods in a horizontal position and what must be its value?

6. A smooth sphere, of weight W, is attached by a string, of length equal to the diameter of the sphere, to a smooth vertical wall against which it rests. Find the tension in the string and the force on the sphere due to the wall.

7. If the wall be tilted through an angle of 30° from the vertical, find the tension and the force between sphere and wall.

8. A rod, whose centre of gravity divides it into two segments, a and b, rests inside a fixed smooth sphere of radius c; find the position of equilibrium and the actions at the points of support.

9. A square figure, ABCD, is formed of four equal uniform rods smoothly jointed together. The system is suspended at A, and is kept in the form of a square by a string joining A and C. Find the tension in the string.

10. The beams of a plain roof are each ten feet long and are connected together by a tie rod fifteen feet long. Find the tension in the rod, assuming that the forces between the walls and beams are vertical, and also find the stress at the ridge.

CHAPTER XII

THE LAWS OF FRICTION AND THEIR APPLICATIONS

THE examples given in the preceding chapter are somewhat artificial, in so far as all surfaces that come into contact are supposed to be perfectly smooth. The laws which rough bodies follow must now be considered. An apparatus by which they can be investigated consists simply of blocks of the material to be experimented upon, which can be drawn over a horizontal board of the same or different material by means of weights placed in a pan attached to a cord passing over a pulley as shown in Fig. 79. The board is there shown

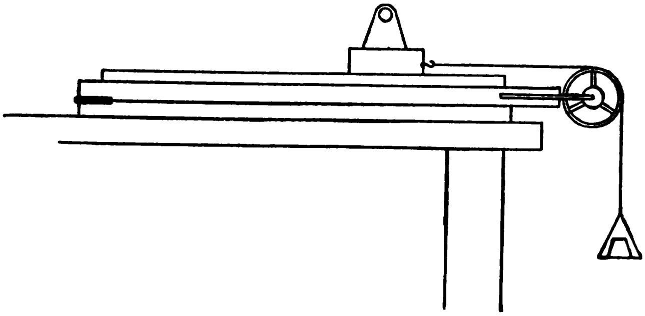


Fig. 79.

placed on an inclinable stand in which the pulley is mounted.

The horizontal force pulling the block consists of the weight of the pan together with the weights in it. If these are absent the action R between table and block must be vertical and equal numerically to W, the weight of the block. Let now a small horizontal force, P, be applied; if sufficiently small no motion is produced. Since equilibrium exists the force due to the plane must be inclined backwards for it is equal but opposite to the resultant of P and W. Its normal component, N, balances W; its horizontal component, F, balances P. F is called the *friction* between the two surfaces. Let now the force P be gradually increased; as long as no motion ensues F must increase at the same rate; it follows that the

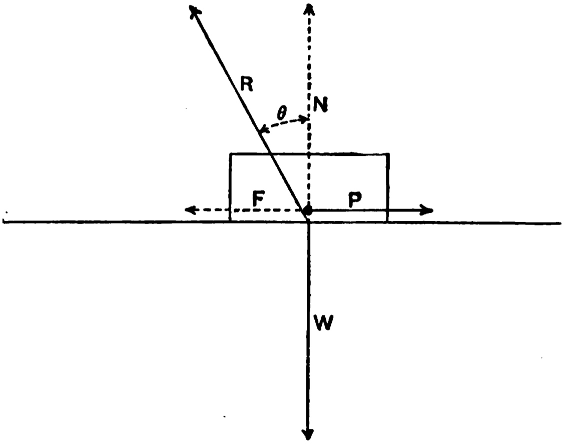


Fig. 80.

friction between the same two surfaces at rest is a variable quantity, its value depending upon the other forces acting upon the movable body. If, however, the increase in P be continued, a value is at last reached for which motion begins, and any further increase in P only goes to increase the motion. From this point onwards F remains practically constant, though there is evidence of a small *decrease* just as motion begins. The greatest value of P, which fails to produce motion, measures the *limiting friction* (F_m). The angle which the resultant R then makes with the normal is called the *limiting angle of friction.* These results can be represented in a diagram by plotting F against P. As long as no motion takes place the curve is a straight line at 45° to the axes. The friction during motion—distinguished by being called *kinetic* friction—is shown a little less than the limiting friction.

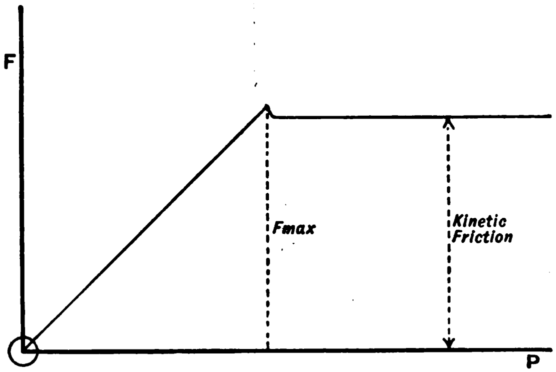


Fig. 81.

Let loads be now placed on the block so as to increase W, and therefore N, the normal pressure between the two surfaces, and for each load let the limiting friction be determined. It will be found that F_m for the same two surfaces is nearly proportional to N,—*i.e.* that F_m/N is nearly a constant. This constant is called the *co-efficient of friction*, and is denoted by μ. Since F_m/N is the tangent of the limiting angle of friction (α), $\mu = \tan \alpha$.

Experiments on a large number of surfaces show that μ is nearly independent of the area of contact, but depends only upon the materials and their state of polish.

The values of various materials varies *from about* $\frac{1}{4}$ *for oak sliding on oak* to $\frac{1}{8}$ *for polished steel on steel.* If the wood surfaces are rubbed over with black lead the value is much reduced. Such a friction-reducing substance is known as a *lubricant.* It acts by separating the surfaces from one another by means of a very friable material which easily crumbles down when motion takes place.

Friction always *opposes* the motion which tends to take, or is actually taking place. Thus positive work is never done by it.

We shall now solve problems in which rough surfaces take their part.

1. A block resting upon a rough inclined plane. The weight acts downwards through the centre of gravity. The reaction, R, of the plane is the

only other force; it must therefore act vertically upwards through the centre of gravity, and be equal to W. It therefore makes an angle θ = the angle of the plane with the normal to the inclined surface. If the angle is greater than α (the limiting angle), equilibrium is not possible. Hence α and thence μ can be determined experimentally by finding the greatest inclination of the plane for which equilibrium exists; μ is the tangent of this angle. The hinged board in Fig. 79 is for this purpose. The block may be loaded through a wide range with very little influence upon the limiting angle. This angle is often known as the *angle of repose*, in consequence of this mode of obtaining it.

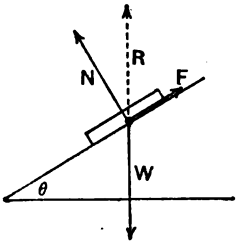


Fig. 82.

The action, R, can of course be resolved into normal and tangential components, as shown in Fig. 82; their values are

$$N = W \cos \theta,$$
$$F = W \sin \theta.$$

These values are ascertained by utilising the first condition of equilibrium.

2. Force required to keep a body at rest on an inclined plane. Let the force P be as shown (Fig. 83). Then we must have

$$P = F + W \sin \theta,$$
$$\text{and} \quad N = W \cos \theta,$$

if the body just fails to move upward. But we have seen that F may have any value between F_m pointing up along the plane if the block is about to slip down, and F_m, pointing down along if it is on the point of being drawn upwards. *There consequently will be a range of possible values for P.* Now $F_m = \mu N = \mu W \cos\theta$; hence all we can say without further information is that

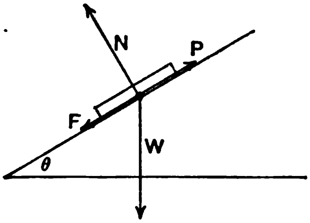


Fig. 83.

$$P > W(\sin\theta - \mu\cos\theta)$$
$$\text{and} \qquad < W(\sin\theta + \mu\cos\theta),$$

where F has been put equal to $\mu N = \mu W \cos\theta$.

One of the possible values makes F zero; the value of P is then the same as for a perfectly smooth surface.

It is a characteristic of all problems in friction that there is a *range* in the possible values of the other forces required to preserve equilibrium, instead of single values.

If, however, we are told in addition that motion is about to take place in any direction, this is tantamount to saying that friction has risen to its maximum value in the opposite direction; the solution of the problem is then quite definite.

If the force P is horizontal the equations become

$$P\cos\theta = W\sin\theta + F.$$
$$N = W\cos\theta + P\sin\theta,$$

Here again F may vary between

$$\pm \mu N, \text{ that is } \pm \mu(W \cos\theta + P \sin\theta).$$

$$\therefore\ P\cos\theta = W\sin\theta \pm \mu(W\cos\theta + P\sin\theta).$$

Selecting the upper sign,

$$P = W\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right).$$

Selecting the lower sign,

$$P = W\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right).$$

These are extreme values; every intermediate value will also preserve equilibrium.

Remembering that $\mu = \tan\alpha$, and dividing both numerator and denominator by $\cos\theta$, we obtain

$$P_{\max} = W\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = W\tan(\theta + \alpha).$$

$$P_{\min} = W\frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = W\tan(\theta - \alpha).$$

If θ is less than the limiting angle of friction, the least (algebraic) value of P—given by the last equation—is negative; in other words, it acts horizontally from right to left. This implies that the maximum upward friction will be called into play only if the force P acts in this way.

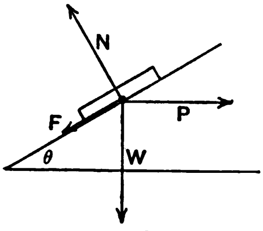


Fig. 84.

Any force intermediate between these extreme values will maintain equilibrium.

The question may be asked, What is the direction of the least value of P that will draw the body up the plane?

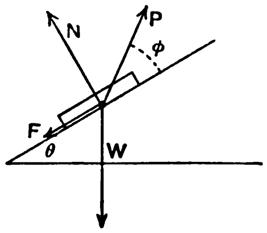


Fig. 85.

Let the angle it makes with the plane be ϕ (Fig. 85).

The resultant R of μN and N makes an angle a with N; resolve the forces perpendicularly to this resultant.

$$P \sin (90^\circ + a - \phi) - W \sin (\theta + a) = 0,$$

$$\text{or} \quad P = \frac{W \sin (\theta + a)}{\cos (a - \phi)}.$$

Now W, θ, and a are constants; thus the numerator is constant. The smallest value of P will correspond to the biggest value of $\cos (a - \phi)$, and this occurs when the cosine is unity, and therefore $\phi = a$.

Therefore to draw a body up an inclined plane most easily, the force should make an angle with the plane equal to the limiting angle of friction. Hence this angle is also known as the *angle of best traction.* It applies of course to a horizontal surface ($\theta = o$), as well as to an inclined plane.

The general way in which this best angle arises can be seen by noticing that the normal component of P diminishes N, and consequently diminishes the friction F. At the same time, however, the component which is effective in producing motion, *viz.* P cos ϕ, diminishes if P remains unchanged. As ϕ increases from zero, the decrease of F, which depends on sin ϕ, is at first greater than the decrease in P cos ϕ. Hence P must at the same time be decreased, or the forces will produce acceleration.

Horizontal Beam resting on Rough Piers. The difference between this and the case in which the piers are supposed to be smooth is that the forces due to the piers need not be vertical. Let them be inclined at angles θ_1 and θ_2 with the vertical (Fig. 86). The following special conditions must be satisfied.

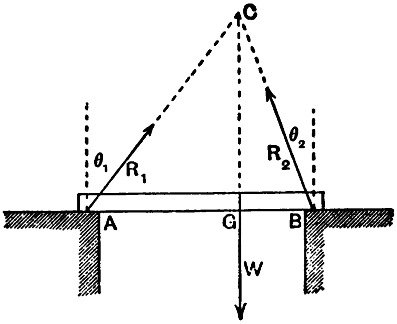


Fig. 86.

i. Neither θ_1 nor θ_2 must be greater than the limiting angle of friction.

ii. The three forces must intersect in one point, such as C.

$$\text{Now}\qquad \overline{\text{CG}} = a \cot \theta_1 = b \cot \theta_2,$$
$$\therefore\quad a \tan \theta_2 = b \tan \theta_1.$$

Hence this relation makes the moment sum vanish about C.

Taking moments about A,

$$\text{R}_2(a + b) \cos \theta_2 = \text{W}a.$$

Taking moments about B,

$$R_1(a + b) \cos \theta_1 = Wb.$$

These are all the equations given by the conditions of equilibrium—three equations containing four unknown quantities, R_1, R_2, θ_1, θ_2. Hence, without further information, the problem is not soluble; or rather, there are an indefinite number of solutions. This further information may be obtained if we know how the beam has reached its equilibrium position. Thus, if after the beam were placed in position the support A were drawn from right to left *so as to slip under the beam*, θ_1 will be the limiting angle which may be supposed known; the other three quantities may then be obtained from the three equations. The student should again notice that in the problem as given there is a *range of possible values.*

If we assume θ_1 to be known, the other quantities become

$$R_1 = \frac{Wb}{(a + b) \cos \theta_1}$$

from the third equation;

$$\tan \theta_2 = \frac{b}{a} \tan \theta_1$$

from the first, whence

$$\cos \theta_2 = \frac{a}{\sqrt{a^2 + b^2 \tan^2 \theta_1}};$$

whence from the second equation

$$R_2 = \frac{W \sqrt{a^2 + b^2 \tan^2 \theta_1}}{(a + b)}.$$

In every case the sum of the horizontal components of R_1 and R_2 must be zero, and the sum of the vertical components must be equal and opposite to W.

Rough and Rigid Joints. When the hinge or hinge pin joining two beams is rough the force between the pin and either of the beams does not in general pass through the pin itself. Consider the special case in which slipping is on the point of taking place. Whatever the extent of contact between the pin and beam may be, each component of the total force between them must be inclined at an angle a with the normal to the pin. Each component will therefore have a moment about the centre of the pin; and since each will be inclined the same way, *i.e.* opposite to the impending motion, their moments will all be of the same sign, and consequently their resultant will also have a moment about their centre and its line of action cannot pass through it. If slipping is not about to take place the resultant *may* pass through the centre, but it is very unlikely to do so.

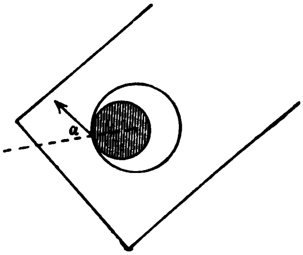


Fig. 87.

Similarly, when two beams are rigidly fastened together the action due to the joint must not be expected to pass through the joint itself. As an example of this we will take the following case.

A pair of compasses rest on a *smooth* cylindrical peg. Find the moment about the joint of the forces due to the joint (Fig. 88).

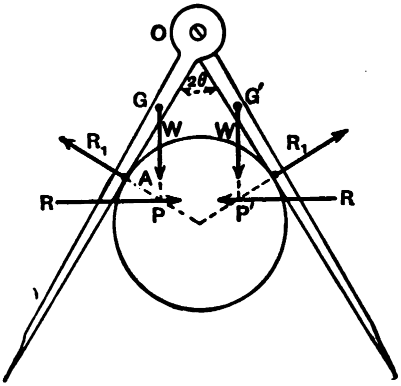


Fig 88.

In this case the force R_1 due to the cylinder is perpendicular thereto, and the weight, W, of each leg acts through G (or G′) the centre of gravity of the leg. The third force, *viz.* the resultant action of the hinge must therefore pass through P, the point of intersection of R_1 and W. Again, by the law of mutual action, this force on the left limb due to the right limb is equal and opposite to that on the right limb due to the left; but symmetry requires that the direction of each of these forces shall be that of the other reversed about the line of symmetry, which is vertical. Hence they must be equal and opposite *horizontal* forces passing through P (or P′). Let the angle between the limbs be 2θ, $OG = l$, and the radius of the cylinder equal r. Then

$$R_1 \sin \theta = W$$

$$R_1 \cos \theta = R$$

$$\therefore R = \frac{W}{\tan \theta} \text{ and } R_1 = \frac{W}{\sin \theta}.$$

The vertical distance, *d*, of R from the hinge is

$$d = \overline{\mathrm{OG}}\cos\theta + \overline{\mathrm{PG}} = \overline{\mathrm{OG}}\cos\theta + \frac{\mathrm{AG}}{\cos\theta}$$

where A is the point of contact with the cylinder;

$$\text{but } \overline{\mathrm{AG}} = \overline{\mathrm{OA}} - \overline{\mathrm{OG}} = \frac{r}{\tan\theta} - \mathrm{OG}$$

$$\therefore d = l\cos\theta + \frac{r}{\sin\theta} - \frac{l}{\cos\theta}$$

$$= \frac{r}{\sin\theta} - l\frac{\sin^2\theta}{\cos\theta}.$$

Therefore the moment of R about O is

$$\mathrm{R}d = \frac{\mathrm{W}}{\tan\theta}\left(\frac{r}{\sin\theta} - \frac{l\sin^2\theta}{\cos\theta}\right) = \mathrm{W}\left(\frac{r\cos\theta}{\sin^2\theta} - l\sin\theta\right).$$

Or it might be found as follows: Since there is equilibrium the moment of R must be equal and opposite to the algebraic sum of the moments of W and R_1; *i.e.* it must equal the negative of

$$\mathrm{W}\,.\,\overline{\mathrm{OG}}\sin\theta - \mathrm{R}_1\,.\,\overline{\mathrm{OA}}$$

$$\text{or } \mathrm{W}l\sin\theta - \frac{\mathrm{W}}{\sin\theta}\cdot\frac{r}{\tan\theta}$$

$$= \mathrm{W}\left(l\sin\theta - \frac{r\cos\theta}{\sin^2\theta}\right).$$

This with sign reversed is the moment required.

If the peg were rough the problem would be indeterminate, unless we were given some additional datum, such as that each limb was on the point of slipping over it. This would fix the direction of R_1 and once more make a single solution the only one possible.

Rolling Friction.—When a substance in the form of a cylinder *rolls* over a surface the resistance to motion is much less than for sliding. This is indicated by the conspicuous advantage shown by wheeled carts over sledges. The action of rolling friction cannot be represented by a single force at the surface of contact; in part it is in the nature of a couple. This statement implies that the single resisting force necessary to account for the motion of the centre of mass of the body will not account for the rotation unless it is placed elsewhere than at the surface of contact. It is perhaps simpler to consider it as acting at this surface and to introduce the necessary opposing couple as well, as we may do in accordance with the fourth theorem on p. 157.

When the rolling takes place over a soft material such as indiarubber the rubber becomes visibly deformed. If vertical equidistant lines are drawn on the edge of the indiarubber and the cylinder extends to this edge the deformation of these lines is somewhat as shown in Fig. 89, which is from a paper by Professor

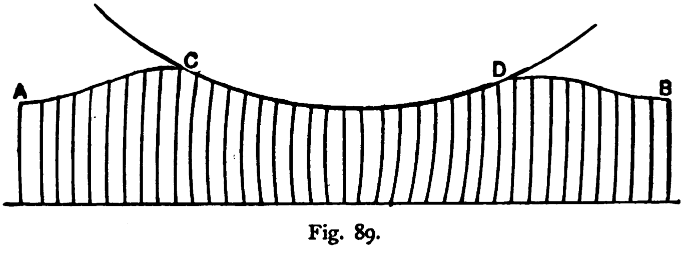


Fig. 89.

Osborne Reynolds. Since the part in contact with the cylinder changes length there must be slip between them. The motion of the wheel is from left to right.

When the ground is hard the action is of the same kind but in a less degree.

The laws propounded by Coulomb for this kind of friction were that it is proportional to the weight of the cylinder and inversely proportional to its diameter.

Friction during Motion. When motion is actually taking place the maximum friction is called into play, or at any rate very nearly so; we can therefore put $F = \mu N$ where μ is not much different from its static value. Thus, to find the acceleration produced by a horizontal force, P, in the motion of a body of mass, *m*, on a rough, horizontal plane, we have

$$P - \mu N = ma$$

$$\text{and} \quad N = \text{the weight} = mg,$$

$$\therefore \quad ma = P - \mu mg$$

$$\text{or} \quad a = \frac{P}{m} - \mu g,$$

in other words it is only the excess of P over friction which produces acceleration.

In the case of a train a distinction must be made between the friction between the driving wheels and rails, and that between the wheels of the carriages and the rails. The former acts *backwards* on the ground and *forwards* on the engine, and is, indeed, the external force which *pushes* the train forward. This is because the lower part of the driving wheels is urged backwards by the forces arising from the steam, and friction resists this backward motion, and to do this acts forward on the wheel. On the other hand the carriages are dragged forward and friction opposes this motion, and to do so must act backwards upon their wheels. When the train is moving with constant velocity on the horizontal these two frictions must be equal and opposite, for they are the only external forces on the train which have a horizontal direction. It is thus clear that, while the friction on the carriage wheels is a hindrance to motion, that on the driving wheels is a necessity without which no motion would occur. When the rails are slippery with rain or ice the driving wheels tend to skid instead of advancing, and any such slip involves a loss of power.

EXAMPLES TO CHAPTER XII.

(1) Find the least horizontal force required to draw a kilogram weight along a rough horizontal plan ($\mu = \frac{1}{2}$).

(2) What acceleration will a force equal to $\frac{2}{3}$ the weight produce in the above case?

(3) Find the least horizontal force required to draw a 10 lb. weight up an inclined plane of 30° angle, the co-efficient of friction being one-third?

(4) What mass will a force of 10,000 dynes acting parallel to a plane of 30° angle keep from slipping down the plane if $\mu = \frac{1}{4}$?

(5) Find in direction and magnitude the least force which will draw a ton weight along a horizontal ground for which $\mu = \frac{1}{2}$?

(6) Find in direction and magnitude the least force which will draw the same weight up a plane of 30° angle for which $\mu = \frac{1}{3}$.

(7) A body is placed on a rough inclined plane whose angle is 45°, and for which $\mu = \frac{1}{3}$; find how long it will take to slide 4 feet from rest.

(8) If it is required to stop a train quickly, should the breaks be applied so as to lock the wheels completely or not?

(9) Does a foul or clean gun kick the most, and why?

(10) A uniform ladder rests against a rough wall ($\mu = \frac{1}{3}$) and upon a rough ground ($\mu = \frac{1}{2}$); what is the greatest inclination to the vertical for which equilibrium can subsist?

(11) If a man equal to twice the weight of the ladder climbs up it, will it be most likely to slip when he is at the bottom or the top? What is the angle with the vertical at which slip begins for both the positions of the man?

(12) The weight of the ladder being 70 lb. find the total force against the wall and against the ground in both cases in problem 11.

(13) A sphere rests against a rough wall and a *rough inclined* plane, the angle between the planes

being 60°; find the general relation between the angles a_1 and a_2 which the forces at the points of contact make with the respective normals.

(14) A uniform bar (length $= l$) rests with one end on a rough horizontal table ($\mu = \frac{1}{3}$) and with the other end tangential to a smooth cylinder of radius r, which is fixed to the same table. What is the least value of the co-efficient of friction between rod and table in order that the rod may not slip? If the cylinder is also rough how is the result modified?

(15) A driving belt transmits power at the rate of 10,000 watts to a pulley of 80 centimetres radius, which rotates at 50 revolutions per minute. What is the total frictional force between the pulley and belt?

(16) A horizontal shaft which passes through a hole near one end of a uniform bar of 10 lb. weight and 4 feet length is rotated at such a rate that the bar maintains a horizontal position. Find the work done by the shaft during 1,000 revolutions.

CHAPTER XIII

MOTION OF A RIGID BODY ABOUT A FIXED POINT

In order to satisfactorily deal with the motion produced when the forces acting are not in equilibrium it is necessary to show how the fact that the turning power of a force depends on its moment can be deduced from the action of forces on particles.

1. Consider a single particle moving in a circular path round a fixed point, O. Let F be the force at any instant, making an angle, θ, with the path. Its moment about O is $Fr \cos \theta$, or $rF \cos \theta$, *i.e.* it is the component force parallel to the path multiplied into the radius.

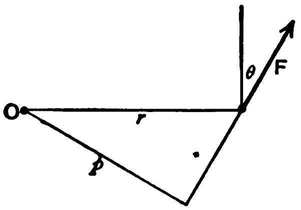


Fig. 90.

Now it is this component which produces the component acceleration a along the path; *i.e.* $F \cos \theta = ma$ where m is the mass. But $a = r\alpha$ where α is the angular acceleration

$$\therefore \ \mathrm{F}r\cos\theta = mr^2 . \mathbf{a},$$

$$\text{or } \frac{\text{Moment}}{\text{angular acceleration}} = mr^2.$$

This result may be expressed in the following words: The moment of the resultant force on a particle which moves in a circle about a point, O, bears a constant proportion to the angular acceleration produced. This proportion, *viz.* mr^2, is called the moment. of inertia of the particle with respect to the point O.

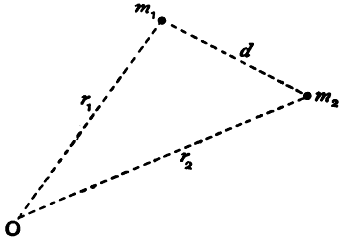


Fig. 91.

2. Let there be two particles rigidly connected together and rotating round a fixed point, O, in the plane of the paper (Fig. 91).

In order to make quite clear the motion that is intended it will be sufficient to point out that r_1, r_2, and the distance between the masses are each to remain constant. Then we have

Moment about O of resultant force on $m_1 = m_1 r_1^2 . \mathbf{a}$

" " " " " $m_2 = m_2 r_2^2 . \mathbf{a}$

for $\mathbf{a}$ is clearly the same for both. Hence Σ(moments about O) $= \mathbf{a}\Sigma(mr^2)$.

Now, the resultant force on m_1 is partly due to

m_2 (*i.e.* it is an internal force), and is partly due to outside bodies. Similarly the force on m_2 is partly due to m_1. But the internal forces not only are equal and opposite, but have the same line of action; their *moments* must therefore be equal and opposite. Hence, in adding the moments those due to internal forces cancel one another; and in consequence the external moments alone need be considered. The equation therefore becomes

$$\Sigma\,(\text{moments of external forces about O}) = \mathbf{a}\Sigma(mr^2).$$

The factor $\Sigma(mr^2)$, which again gives the ratio of the sum of the moments to the angular acceleration is the moment of inertia of the two particles.

Precisely the same remarks will apply, however many particles there may be—the summations being extended to them all—provided that they are rigidly connected together, so that $\mathbf{a}$ is the same for all.

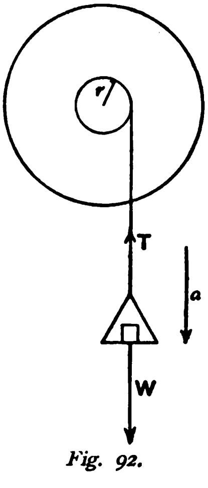


Fig. 92.

Experimental Verification. This law of proportionality can be experimentally verified with a rotating wheel mounted on an axle. Round the axle a thread, supporting a scale pan, is wound. Definite weights are placed in the pan, and when the pan is set free these descend with acceleration and cause the wheel to rotate.

The moment of the forces causing rotation is not that of the weights suspended by the thread, since they have acceleration. Let W be their total weight and a the linear acceleration downwards. The tension in the string is $T = \frac{W}{g}(g - a)$. If r is the radius of the axle the moment of T is Tr; this is the *applied moment.* The acceleration a can be found by measuring the height fallen in a definite time from rest, and calculating by means of the formula $s = \frac{1}{2}at^2$. The angular acceleration is a/r. Make a series of determinations, using different weights, and plot the applied movement in each case against the acceleration produced; the points plotted should all lie near a straight line (Fig. 93). This line will not go through the origin; this arises from the fact that no account has so far been taken of friction. Find the smallest weight that will produce motion; this multiplied into r gives the amount of friction, and will be found approximately equal to the intercept on the vertical axis. The difference between the applied and frictional moment is the resultant (or effective) moment. Hence, if the resultant moment is plotted against the angular acceleration the curve is a straight line passing through the origin, and these two

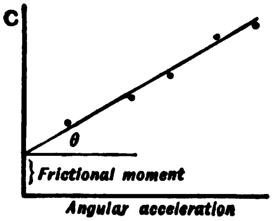


Fig. 93.

quantities are therefore proportional to one another.

The ratio of the two has been defined as the *moment of inertia* of the rotating body. Its value is therefore tan θ (Fig. 93), provided that the moment and the angular acceleration are plotted on the same scale. If unit length on the diagram represents k units of moment and l units of angular acceleration, the moment of inertia is $\frac{k}{l}$ tan θ. We shall denote it by I.

It should be noticed that the moment of a force corresponds to angular acceleration, *not* angular velocity. If the forces have no moment the angular velocity must be constant, but is not necessarily zero.

We have seen that $I = \Sigma(mr^2)$; therefore, theoretically at any rate, it must be possible to calculate it, if the masses and positions of all the particles of the body are known. In most cases, even for regular bodies, the mathematics required is of a higher order than that to which this volume is restricted. In some cases, however, the calculation can be based on the introductory theorems of the differential and integral calculus, and this we proceed to do.

i. A circular ring of radius r rotating about an axis through its centre perpendicular to its plane.

Since each part of it is at the same distance from the axis,

$$\Sigma mr^2 = r^2\,\Sigma m = r^2 \times \text{total mass.}$$

ii. Thin uniform rod rotating about an axis perpendicular to its length and passing through one end.

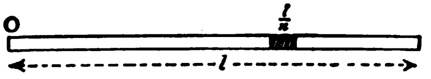


Fig. 94.

Divide it into n equal lengths, each of which is $\frac{l}{n}$; the mass of each is m/n, where m is the mass of the whole rod; the distance of each element from O may be taken as the distance of its more distant end; the error thereby committed becomes vanishingly small when n is made infinitely large. The moment of inertia is the sum of the moments of inertia of the separate parts. Hence, approximately,

$$\begin{aligned} \mathrm{I} &= \frac{m}{n}\left(\frac{l}{n}\right)^2 + \frac{m}{n}\left(\frac{2l}{n}\right)^2 + \frac{m}{n}\left(\frac{3l}{n}\right)^2 + \dots + \frac{m}{n}\left(\frac{nl}{n}\right)^2 \\ &= \frac{ml^2}{n^3}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right). \\ &= \frac{ml^2}{n^3}\ \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Now if n be made infinitely large—in which case the error of the above becomes infinitely small—

$$\mathrm{I} = \frac{ml^2}{n^3} \cdot \frac{n^3}{3} = \frac{ml^2}{3}$$

i.e. the moment of inertia is the same as if the whole body were concentrated at a point distant

$\frac{l}{\sqrt{3}}$ from O; this distance is called the *spin-radius* (or *swing-radius*, or *radius of gyration*[1]) with respect to the particular axis.

iii. A thin uniform rod of length $2l$, with the axis at right angles to it, and passing through its centre.

This is obviously the same as two rods similar to the last placed end to end, with their axes of rotation coinciding; hence

$$I = 2\frac{ml^2}{3} = \text{total mass} \times \frac{l^2}{3}.$$

$$= \text{total mass} \times \frac{(\text{total length})^2}{12}.$$

The spin-radius is the total length $\div 2\sqrt{3}$.

iv. A circular disc (or cylinder) of radius r, mass m, spinning about an axis through its centre and perpendicular to its circular section.

This may be divided into n hollow cylinders of thickness r/n; the mass of each such cylinder is proportional to its radius, say μ times its radius; when n is large each may be supposed concentrated at its outer surface with an error which becomes smaller as n increases and finally vanishes. Hence mass of each × its radius^2 $= \mu \times$ its radius^3, and

$$I = \mu[r_1^3 + r_2^3 + r_3^3 + \text{etc.}], \text{ approximately,}$$

$$= \mu\left[\left(\frac{r}{n}\right)^3 + \left(\frac{2r}{n}\right)^3 + \left(\frac{3r}{n}\right)^3 + \ldots + \left(\frac{nr}{n}\right)^3\right],$$

---

[1] The term gyradius or gyrad is suggested as a suitable name, which might be more acceptable on the Continent.

$$= \mu\frac{r^3}{n^3}\Big[1^3 + 2^3 + 3^3 + 4^3 + \ldots n^3\Big],$$

$$= \frac{\mu r^3}{n^3} \cdot \left\{\frac{n(n+1)}{2}\right\}^2 = \frac{\mu r^3}{n^3} \cdot \frac{n^4}{4} \text{ (when } n = \infty\text{)},$$

$$= \mu n \cdot \frac{r^3}{4}.$$

The total mass $= \mu\frac{r}{n} + \mu\frac{2r}{n} + \ldots \mu\frac{nr}{n}$,

$$= \frac{\mu r}{n}\Big[1 + 2 + 3 + \ldots + n\Big],$$

$$= \frac{\mu r}{n} \cdot \frac{n(n+1)}{2} = \frac{\mu r n}{2} \text{ (when } n = \infty\text{)}.$$

$$\therefore \text{ I} = \text{total mass} \times \frac{r^2}{2};$$

hence the spin-radius is $\frac{r}{\sqrt{2}}$.

These examples will serve as illustrations. Other cases are best solved by means of the calculus, although it is possible to treat them without explicit reference thereto. The important fact to grasp at present is that in all cases the moment of inertia is calculated by dividing the whole body into very small portions, of each of which the moment is known, adding the separate values, and then finding the limiting value approached as each portion diminishes to the vanishing point (Fig. 95).

The unit in terms of which a moment of inertia is expressed depends upon the units chosen for mass and length respectively. Thus, if the mass is one gram and its distance from the axis is one cm., the moment of inertia is one gram-cm^2.

We shall now prove an auxiliary theorem of frequent application.

To calculate the moment of inertia about any axis when the value about a parallel axis through the centre of mass is known (Fig. 95).

Let G be the centre of mass and O the point representing the axis, and let the moments of inertia about the two axes be I_G and I_O respectively. Then $I_O = \Sigma(mr^2)$ and $I_G = \Sigma(mR^2)$, r and R being the distances of any particle, m, from the axes. Let $\overline{GO} = a$; drop a perpendicular from any particle on $\overline{GO}$, meeting it in N. Then $r^2 = R^2 + a^2 - 2a \,.\, \overline{GN}$.

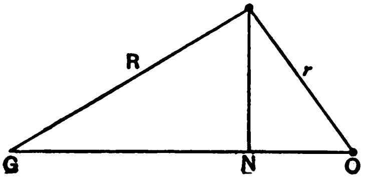


Fig. 95.

Therefore

$$\Sigma(mr^2) = \Sigma(mR^2) + \Sigma(ma^2) - \Sigma\,(2ma\overline{GN}).$$

But a is a constant and $\Sigma(m\overline{GN})$ is the sum of the moments of the masses about the centre of mass, and this is zero; therefore

$$I_O = I_G + a^2 \times \text{total mass}.$$

Example A. Derive iii from ii on p. 206 by the above theorem.

Example B. What force applied at its circumference will stop, in 10 seconds, a grindstone which is spinning once a second and whose radius is 20 centimetres, and whose mass is 25,000 grams?

The moment of inertia is

$$25{,}000 \times 20^2/2 = 5{,}000{,}000 \text{ gram-cm}^2.$$

The angular acceleration is $-\frac{2\pi}{10}\frac{\text{radians}}{\text{sec}^2}$.

The opposing couple is therefore $2\pi \times 5 \times 10^6$ dyne-centimetres, and this is produced by a peripheral force of $\frac{2\pi}{20} \times 5 \times 10^6$ dynes

$$= \frac{\pi \times 10^6}{20} \text{ dynes.}$$

Example C. A string carrying weights passes round a smoothly mounted pulley as in Atwood's machine; find the acceleration if the weights be free to move. Also find the difference of the tensions on the two sides of the pulley.

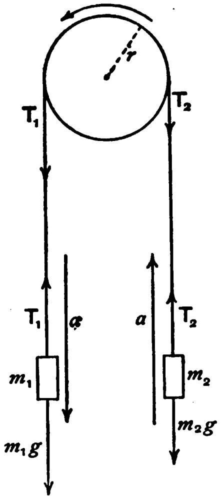


Fig. 96.

Let the masses and forces be as shown (Fig. 96); then if m_1 is more massive than m_2 the acceleration will be downward on the left and upward on the right; let its value be a. Then

$$m_1g - T_1 = m_1a$$

$$T_2 - m_2g = m_2a$$

or

$$T_2 - T_1 + (m_1 - m_2)g = (m_1 + m_2)a$$

14

Now, the wheel is set rotating by the string; if the radius of the wheel is r, the counter-clockwise couple setting it in rotation is $(T_1 - T_2)r$; and the angular acceleration produced is a/r if there is no slipping of the string on the wheel. If Mk_0^2 is the moment of inertia of the wheel (M = mass, k_0 spin-radius) about its centre.

$$(T_1 - T_2)r = Mk_0^2 a/r.$$

Putting the value of $T_2 - T_1$ given by this into the previous equation, we get

$$- Mk_0^2 a/r^2 + (m_1 - m_2)g = (m_1 + m_2)a$$

$$\text{or} \qquad a = \frac{(m_1 - m_2)g}{m_1 + m_2 + Mk_0^2/r^2}$$

Hence, as we said before (p. 56), the effect of the pulley is to introduce an additional term into the denominator without affecting the numerator; Mk_0^2/r^2 is the *effective mass* of the pulley. If this were a solid cylinder $k_0^2/r^2 = \frac{1}{2}$, and in this case the effective mass is half its real mass. For an ordinary wheel the mass is concentrated nearer the rim, and the effective mass is a greater fraction than this. The difference of tensions is

$$T_1 - T_2 = \frac{Mk_0^2 (m_1 - m_2)g}{r^2(m_1 + m_2) + Mk_0^2}.$$

EXAMPLES. CHAPTER XIII.

(1) A grindstone of 3 feet diameter and weighing 250 lb. is set rotating at the rate of one

revolution in two seconds. What average couple is necessary to do this if the period of starting lasts 3 seconds?

(2) The mass of the pulley wheel on an Atwood's machine is 24 oz., and the masses at the end of the string are 32 and 33 oz. respectively. Find the difference in the tension of the cord on the two sides of the pulley, and the couple acting on the wheel during the free movement of the weights? (Radius of pulley = 6 in. Spin radius = 5 in.)

(3) Which will require the greater force (applied at its circumference) to give it a certain angular speed in a given time, a wheel of steel of density 7·8, or a *similar* wheel of aluminium (density 2·6) of twice the linear dimensions?

(4) Find the moment of inertia of a cylinder of 12 lb. mass and 4 inches radius, about an axis parallel to its own, and distant 10 feet from it.

(5) The moment of inertia of a certain body is 22,000 gram-cm.^{2}; what is its value expressed in terms of the pound and foot taken as fundamental units?

(6) Calculate the moment of inertia of a hollow cylinder, about its own axis, the mass being 100 lb., and the external and internal radii being 12 and 8 feet respectively.

(7) Calculate the moment of inertia of a circular disc, of mass 4 grams and radius 8 centimetres, spinning about a tangent to its circumference.

(8) A flywheel of 10 feet diameter, weighing 10 tons, is making 40 revolutions per minute, when thrown out of gear. In what time does it come to rest, if the diameter of the axle is 6 inches and the co-efficient of friction ·05? (Neglect spokes.)

(9) Show that the same result is obtained for the moment of inertia of a rod, as that which is obtained on p. 205, if we assume each of the elements to be concentrated at the end of it which is nearer the axis; and thereby prove that the error committed in either case is negligible when the limiting case is reached.

(10) Treat the case of a disc (p. 206) in a similar way.

CHAPTER XIV

MOTION OF A RIGID BODY

In the previous chapter the body was supposed to rotate about an axis fixed in space as well as fixed in the body. In the present chapter we shall suppose the axis to move, as, for example, when a cartwheel rolls on the ground. Even now only a group of simple cases will be considered, *viz.* those in which the forces and the body are confined to a single plane, or if the body is not wholly in this plane it must at least be symmetrical with respect to it. The reason for this restriction will be alluded to at the end.

We commence with two definitions, which were dispensed with in the previous chapter.

1. The moment of any velocity V about O is the product of V into its perpendicular distance, *p*, from O.

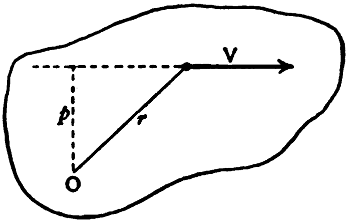


Fig. 97.

2. The moment of momentum of the particle of mass, *m*, whose velocity is V, is the

product of its momentum, mV, into the same perpendicular distance; it is therefore equal to the moment of its velocity multiplied by m.

Since both of these moments are obtained in the same way as the moment of a force, and velocity and momentum are both resolvable into components according to the same law as forces, it follows in the same way as for forces (p. 149) that the moment of each about any axis is the sum of the moments of the components about the same axis.

Now we have seen (p. 145) that every case of motion in this plane can be built up of a translation together with a rotation about the given axis—that is to say, that the velocity of any point can be resolved into two components, one of them parallel to the translation which we shall denote by V, and which is the same for every point in the body; and the other normal to the line joining the axis with the point in question; the latter component is proportional to the length of this line, r, and to the angular velocity, ω; its value is $r\omega$. Let the resultant velocity be v.

Apply these definitions first to the case where the axis is fixed in space as well as in the body. In this case V is absent and the velocity of any particle is $r\omega$ normal to r; its moment is therefore $r \cdot r\omega$ or $r^2\omega$; the moment of momentum is $mr^2 \cdot \omega$. If now the moments of momentum of all the particles be added together the result is $\Sigma(mr^2 . \omega)$, or, since ω is the same for all in the case of a rigid body, $\omega\Sigma(mr^2)$, which is $I\omega$ where I is the moment of inertia with respect to the particular axis. Written in this form which applies only to a rigid body, the total moment of momentum is called the *angular momentum.* Now it was proved that the moment of all the external forces about a fixed axis is equal to $I\mathbf{a}$ where $\mathbf{a}$ is the angular acceleration. But $\mathbf{a}$ is the rate of increase of ω, and since I is a constant $I\mathbf{a}$ must be the rate of increase of $I\omega$. Hence we may say: The sum of the moments, about a fixed axis, of the external forces on a rigid body is equal to the rate of increase of the angular momentum about the same axis.

This statement is merely the expression in fresh terms of the theorem previously proved.

Now in the more general case, for which the axis is itself in motion in space (though still fixed in the body), the total moment of momentum (in virtue of the theorem with respect to the moment of a resultant) can be found by adding algebraically to the moment above found that due to the translational component of velocity. Let this component be V, and remember that it is the same for every particle. The moment of momentum for any one particle is mVp (cf. Fig. 97), and the total moment of momentum due to the translation is therefore $\Sigma(mVp)$, or, since V is the same for all, $V\Sigma(mp)$.

But $\Sigma(mp)$ is the moment of all the *masses* with respect to a line through O parallel to V; and we know that this is permanently zero if, and not unless, the line passes through the centre of mass. Hence, *if the axis passes through the centre of mass* the contribution of the translation to the total moment of momentum is permanently zero, and so is its rate of change; hence, the latter is still equal to the rate of change of angular momentum, which is again, therefore, equal to the total moment of the forces. In the same way, if the translation is in a curve it is shown that nothing is contributed by the central acceleration. Hence the equations of motion of our rigid body in one plane are

$$\text{Linear acceleration of centre of mass} = \left\{\frac{\text{Vector sum of the external forces}}{\text{Total mass.}}\right\}$$

$$\text{Angular acceleration} = \left\{\frac{\text{Algebraic sum of the moments of the external forces }\textbf{about the centre of mass}}{\text{Moment of inertia about the centre of mass.}}\right\}$$

N.B.—Unless the moving axis of rotation passes through the centre of mass, the angular acceleration **cannot** be obtained by dividing the sum of the moments of the external forces, with respect to the axis, by the moment of inertia about the same axis. The rate of increase of the sum of the moments of momentum arising from the translational components of velocity would first require to be subtracted from the numerator. It may be noted that

$$\Sigma(mp) = \mathrm{P}\Sigma(m),$$

where P is the distance between the axis and the centre of mass measured perpendicularly to V. Since P varies as the centre of mass moves round the axis, the amount to be subtracted is also variable and the calculation is complicated in consequence.

When the plane containing the forces is not also a plane of symmetry, the problem of finding the motion is too difficult to be treated here. In order to indicate where the difficulty arises, consider the case in which a certain force acts transversely through the centre of mass of a long cylinder, while an equal opposite force acts at some point of the circumference which does not lie in the plane of symmetry. About what axis will the cylinder begin to turn? The forces have a moment about the longitudinal axis, and also about a transverse one through the centre of mass. If the angular acceleration about each of the axes were the same, it can be shown that the resultant rotation would be about an axis perpendicular to the plane containing the couple; for the relation between the component accelerations would be the same as between the component moments producing them. But this is not the case. The body is more easily set in rotation about one axis than about another; thus, about the longitudinal axis, motion is produced more easily owing to the small moment of inertia with respect to this axis; on the other hand, motion about the transverse axis is generated only with difficulty. The consequence is that the resultant motion more nearly consists in a rotation round the longitudinal axis than might at first be expected.

It can be shown that there are always three **principal axes,** the rotation produced about each of which can be found by the rules given in this chapter; the actual motion consists in the superposition of these three motions.

This complication is excluded from the cases dealt with in this chapter, in all of which the forces have a moment about one principal axis only—*viz.* that which is perpendicular to the plane of symmetry.

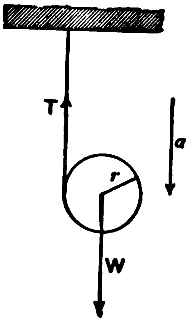


Fig. 98.

ILLUSTRATIVE EXAMPLES.

1. A reel is allowed to fall, the thread being attached to a fixed support. Supposing the thread keeps tight and vertical throughout, calculate the acceleration of the centre of mass of the reel (Fig. 98).

We shall suppose the reel to be fully wound, and further that the amount of thread removed during the fall is so small that the mass of the reel remains sensibly constant. The external forces are, W the weight and T the tension in the thread.

$$\therefore \mathrm{W} - \mathrm{T} = ma \quad . \; . \; . \quad [1]$$

where a is the acceleration acquired. Now the reel can only fall by unwinding, and its angular acceleration $\mathbf{a} = a/r$ where r is its radius. The moment of the external forces about the centre of mass is Tr (for W has no moment); the moment of inertia $= \frac{1}{2}mr^2$, if we can treat the reel as a homogeneous cylinder. Therefore

$$\mathbf{a} = \frac{\mathrm{T}r}{\frac{1}{2}mr^2} = \frac{2\mathrm{T}}{mr},$$

$$\text{hence } a = \frac{2\mathrm{T}}{m} \quad . \; . \; . \quad [2]$$

[1] and [2] suffice to calculate both a and T.

$$\mathrm{T} = \mathrm{W} - ma = \mathrm{W} - 2\mathrm{T},$$

$$\mathrm{T} = \frac{\mathrm{W}}{3}.$$

$$\therefore a = \frac{2\mathrm{W}}{3m} = \frac{2}{3}g.$$

The angular acceleration is $\frac{2g}{3r}$.

2. A cylinder rolls, without slipping, down a rough inclined plane. Calculate the acceleration (Fig. 99).

Here the forces are W, the normal pressure N, and the friction F which acts up the plane. Hence

$$\mathrm{W} \sin \theta - \mathrm{F} = ma,$$

a being acceleration parallel to the plane.

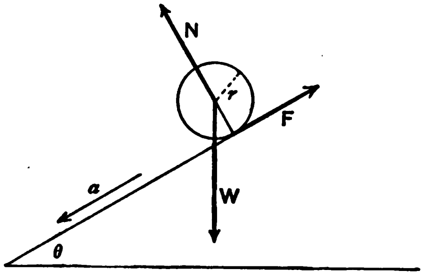


Fig. 99.

[F must not be put equal to μN, for we do not know if the cylinder is on the point of slipping. The friction is supposed to be so great that slipping does not take place, but how far it is from slipping is not known.]

The moment of the forces about the centre of mass is Fr, and since there is no slip $a = \mathbf{a}r$, hence

$$\mathbf{a} = \frac{Fr}{\frac{1}{2}mr^2},$$

$$F = \frac{ma}{2},$$

$$\text{and } W \sin\theta - \frac{ma}{2} = ma;$$

$$\text{or } a = \frac{2W \sin\theta}{3m} = \frac{2}{3} g \sin\theta,$$

i.e. the acceleration is uniform, and 2/3 the value that it would be if the plane were perfectly smooth.

Knowing a the friction can be calculated from the equation $F = \frac{ma}{2}$; whence $F = \frac{W \sin\theta}{3}$.

Since there is no motion perpendicular to the plane,

$$N = W \cos\theta,$$

therefore $$\frac{F}{N} = \frac{1}{3} \tan\theta,$$

Now $\frac{F}{N}$ cannot be greater than μ, the co-efficient of friction, hence if $\tan\theta > 3\mu$, a simple rolling motion will not take place, but will be accompanied by slipping. In this case we may assume $F = \mu N$, and the equations become

$$W \sin\theta - \mu N = ma$$
$$N = W \cos\theta,$$

whence $a = \frac{W}{m}(\sin\theta - \mu \sin\theta) = g(\sin\theta - \mu\cos\theta)$, the same value as if there were translation only. But besides the translation there is angular acceleration

$$a = \frac{Fr}{I} = \frac{\mu N r}{I} = \frac{\mu r W \cos\theta}{I}.$$

The case of "translation only" can be brought about by inserting a "slipper" beneath the cylinder, as in scotching a wheel in going down hill.

If the cylinder is replaced by a sphere the moment of inertia is $\frac{2}{5}mr^2$; putting this instead of $\frac{1}{2}mr^2$ we get $a = \frac{5}{7}g\sin\theta$. Therefore the sphere would roll the more quickly of the two, if both start together.

EXAMPLES.

1. Which will roll the more quickly down an inclined plane—a cylinder or a sphere? Compare their accelerations.

2. A reel of cotton is drawn along a rough horizontal table by one end of the thread, which is pulled horizontally with a tension T. Find the acceleration, assuming the reel to be a homogeneous cylinder of constant mass, m, and rim-radius r, the radius of the wound portion being r'. (Assume no slip.)

3. A pulley wheel lies in a notch cut in a rough horizontal table, the axle resting on the table itself. If a thread be wound round the rim of the wheel, and this thread be pulled horizontally, what will be the direction of motion, the thread being supposed to leave the wheel on the under side?

4. Find the acceleration in (3) from the following data:—Diameter of the wheel 6 cm., of the axle 1 cm., mass of wheel 100 grams, its spin-radius 2 cm., and the tension in the string 1000 dynes.

5. At what angle to the horizontal must the thread of a partially unwound reel be pulled, in order that the reel may move uniformly without rotation?

6. What is the least value of the co-efficient of friction between a wheel and the ground, in order that the wheels of a loaded cart which weighs in the aggregate 12 cwt., may move uniformly without slip on horizontal ground, the friction at the axles being together equivalent to a force of 1 cwt. acting 2 in. from the centre of the axle, and the radius of the wheel being 2 ft.?

7. Assuming that slip just fails to occur, what acceleration will a horizontal force of 120 lb. weight produce, moment of inertia of each of the four wheels being 120 lb.-ft.^{2}?

8. What will be the effect on the motion of the cart if the ground becomes (1) rougher, (2) smoother, than in this critical case?

CHAPTER XV

ENERGY OF A RIGID BODY

THE kinetic energy of a rigid body is the sum of the kinetic energies of its separate parts. We shall now find concise expressions for it.

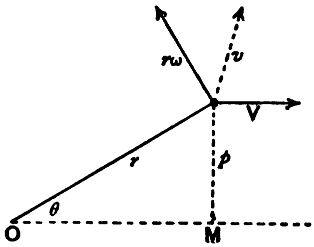


Fig. 100.

Kinetic Energy. The velocity of any particle being v its kinetic energy is $\frac{1}{2}mv^2$. Now it has been shown that v can be resolved into two components representing a translation V, together with a velocity $r\omega$ due to rotation (Fig. 100). If θ is the angle shown in the figure

$$v^2 = V^2 + r^2\omega^2 - 2Vr\omega\cos\theta.$$

Now V and ω are the same for all the particles, and $r\cos\theta$, which may be written p, is the distance between the axis and the particle measured perpendicularly to V.

Therefore the kinetic energy of the whole body is

$$\Sigma(\tfrac{1}{2}mv^2) = \Sigma(\tfrac{1}{2}mV^2) + \Sigma(\tfrac{1}{2}mr^2\omega^2) - (mV\omega p).$$

Now if the axis of rotation passes through the centre of mass $\Sigma(mp) = 0$. Hence, the total kinetic energy of a rigid body moving parallel to one plane can be expressed as the sum of two parts: 1st the kinetic energy of the body supposed concentrated at and moving with the centre of mass; 2nd, a part due to rotation and equal to $\Sigma(\frac{1}{2}mr^2\omega^2) = \frac{1}{2}I\omega^2$, where I is the moment of inertia with respect to an axis passing through the centre of mass perpendicularly to the plane in question, and ω is the angular velocity. If V is zero the term $\Sigma(mV\omega p)$ is zero whether the axis passes through the centre of mass or not. Hence, if a rigid body rotates about any axis *fixed in space and in the body*, its kinetic energy is $\frac{1}{2}I\omega^2$ where I is the moment of inertia about the axis in question.

That these expressions are a measure of kinetic energy can also be shown in an alternative way. If a couple be applied to a rigid body so as to oppose the rotation, this rotation will gradually be reduced to zero. Let C be the couple and θ the angle turned through while the body ceases rotating. The work done by the couple can be calculated by replacing it by its two equivalent forces, F, at a distance apart $(r_1 + r_2)$ (Fig. 101), where r_1 and r_2 are the perpendicular distances of these forces from any axis, O. In a rotation θ, which is so small that the motion of

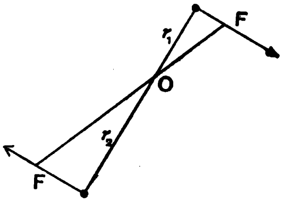


Fig. 101.

the points of application of the forces may be considered parallel to the forces, the work done is

$$\begin{aligned} \mathrm{F}r_1\theta + \mathrm{F}r_2\theta &= \mathrm{F}(r_1 + r_2)\theta \\ &= \text{moment of couple} \times \theta. \end{aligned}$$

It follows that, if the moment is constant, the work done is proportional to the angle, however large this is; or, the work done by a constant couple is equal to its moment multiplied into the angle the body turns through.

This is true whether the body is simultaneously undergoing translation or not, for it is clear that as regards translation the points of application of the forces move each through the same distance in the direction of the forces; the amounts of work done by them are therefore equal, but opposite, and add up to zero.

Now, if the initial angular velocity be ω, and $\mathbf{a}$ be the angular acceleration, the body will cease rotating in a time $\omega/\mathbf{a}$, and the angle θ turned through in this time is

$$\tfrac{1}{2}\omega \times \text{time, or } \theta = \tfrac{1}{2}\frac{\omega^2}{\mathbf{a}}.$$

Again $\mathbf{a} = \mathrm{C}/\mathrm{I}$, where I is the moment of inertia about the centre of mass, hence

$$\text{Work done by couple} = \mathrm{C}\theta = \tfrac{1}{2}\mathrm{I}\omega^2$$

which, therefore, represents the kinetic energy due to rotation. Since the translation of the body can be destroyed by the application of a single force through the centre of mass as though the whole mass were gathered into a particle there, the kinetic energy due to translation is the same as that of this particle—*viz.* $\frac{1}{2}mV^2$ where m is its total mass and V its velocity of translation. Hence, as before, the the total work that the body can do while stopping is $\frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$, and this is, therefore, its total kinetic energy.

Potential Energy due to Gravity. Let a group of particles of weights w_1, w_2, w_3, etc., be initially at vertical heights h_1, h_2, h_3, etc., above the ground. If these heights become h_1', h_2' h_3', etc., the work done against gravity is:

$$w_1(h_1' - h_1) + w_2(h_2' - h_2) + w_3(h_3' - h_3), \text{ etc.}$$

Let H, H′, be the initial and final position of the centre of gravity, then

$$H = \frac{w_1h_1 + w_2h_2 + \text{etc.}}{w_1 + w_2 + \text{etc.}}$$

$$H' = \frac{w_1h_1' + w_2h_2' + \text{etc.}}{w_1 + w_2 + \text{etc.}}.$$

Hence the increase in potential energy to gravity is

$$(H' - H)\Sigma(w) = W(H' - H),$$

where W is the total weight.

We see from this formula that the increase is the same as if the whole body were concentrated at and moving with the centre of gravity.

N.B.—This result is true as far as the energy *due to gravity* is concerned whether the body is rigid or not; but, if the particles exert force one on another, additional energy may arise from this cause if they change their *relative* positions; no such increment due to internal work can arise if the body is rigid.

Example. Problem ii. (p. 219).

Decrease of potential energy while the cylinder moves a distance l along the plane is $Wl \sin\theta$. The work done against F is *zero*, for there is no slip; hence $Wl \sin\theta$ must equal the increase of kinetic energy, or

$$\tfrac{1}{2}mV^2 + \tfrac{1}{2}I\omega^2 = Wl \sin\theta$$

$$\text{or} \qquad \tfrac{1}{2}mV^2\left(1 + \frac{I}{m}\cdot\frac{\omega^2}{V^2}\right) = Wl \sin\theta.$$

$$\text{Now} \qquad V = r\omega \text{ and } I = \tfrac{1}{2}mr^2$$

$$\therefore \quad \tfrac{1}{2}mV^2\left(1 + \frac{r^2}{2}\cdot\frac{1}{r^2}\right) = Wl \sin\theta$$

$$V^2 = \frac{4gl \sin\theta}{3}.$$

Comparing this with the standard equation $V^2 - V_0^2 = 2as$ we see that the acceleration is $\tfrac{2}{3}g \sin\theta$.

Compound Pendulum. This consists of a rigid body suspended at any point so that it may swing like a pendulum. From its deflected position (Fig. 102) the centre of mass, G, descends a vertical height,

$$H_m = b(1 - \cos\theta) = 2b \sin^2\frac{\theta}{2} \text{ (Fig. 102)}$$

to its lowest position. The force at O does no work; hence

$$\tfrac{1}{2}I_O\omega^2 = 2Wb \sin^2\frac{\theta}{2}$$

where I_O is the moment of inertia about O; and ω is the angular velocity when G is at M. If V is the velocity of G at its lowest point

$$V = b\omega \quad \therefore$$

$$\therefore \quad V^2 = \frac{4Wb^3}{I_O}\sin^2\frac{\theta}{2}.$$

If the pendulum is a uniform bar and O is at one end

$$I_O = \frac{m}{3}(2b)^2 = 4\frac{mb^2}{3},$$

$$\therefore \quad V^2 = 3gb\sin^2\frac{\theta}{2}.$$

The student should verify that he may write the kinetic energy,

$$\tfrac{1}{2}mV^2 + \tfrac{1}{2}I_G\omega^2,$$

where I_G is reckoned with respect to the centre of mass.

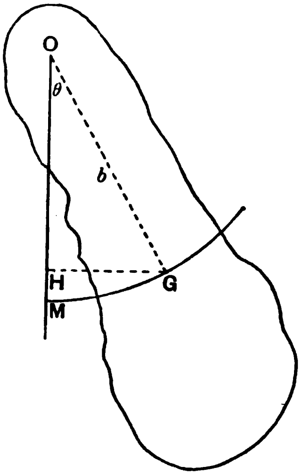


Fig. 102.

If the body had been suspended at its centre of mass by a string of length *b*, fastened to a fixed point O, the body would have descended from rest *without rotating*; because neither the tension in the string nor the weight would then have any moment about G. The velocity of the centre of mass would then have been the same as for a simple pendulum of length b—*i.e.*

$$V^2 = 4gb\sin^2\frac{\theta}{2};$$

whence $$\tfrac{1}{2}mV^2 = 2Wb\sin^2\frac{\theta}{2}.$$

Thus the total kinetic energy gained is the same in both cases, but it is differently distributed.

Oscillating Weights. A string carrying equal weights, W, at its ends passes over two horizontal smooth pegs distant $2a$ one from the other; if a small weight be attached midway between the pegs find how low it will sink before coming to instantaneous rest if $w < 2W$ (Fig. 103).

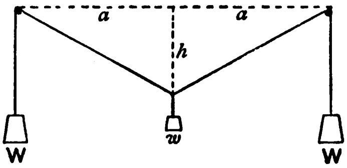


Fig. 103.

When the weight w has descended a distance h the others have risen a distance $\sqrt{a^2 + h^2} - a$. The total work done by gravity during this motion is $wh - 2W(\sqrt{a^2 + h^2} - a)$.

No work is done by the pegs during the motion.

But since the system is at rest both initially and finally the change in the kinetic energy is zero. Hence work done is zero or

$$wh + 2Wa = 2W\sqrt{a^2 + h^2}.$$

Square:

$$w^2h^2 + 4W^2a^2 + 4wWah = 4W^2(a^2 + h^2)$$
$$\text{or } h[(w^2 - 4W^2)h + 4wWa] = 0.$$
$$\text{Whence } h = 0, \text{ or } h = \frac{4wWa}{4W^2 - w^2}.$$

The second solution is the one sought for. But the former ($h = 0$) is interesting as indicating that if w returns towards its starting point the whole system will be again at rest when it reaches it. Neither value, it should be realised, gives a position of equilibrium or permanent rest. There is an upward force on w when at its lowest point, and a downward force when at its highest point; these cause it to move backwards and forwards continually on its path. The position of equilibrium into which the friction inevitably present in reality would cause it finally to settle down is nearly half way between 0 and h provided w/W is small. The student will easily verify that it is in a position given by

$$\frac{2h'}{\sqrt{a^2 + h'^2}} = \frac{w}{W}$$
$$\text{or } h' = \sqrt{\frac{w^2a^2}{4W^2 - w^2}}$$

which, if w/W is negligible, is $\frac{wa}{2W}$, whereas with the same restriction $h = \frac{wa}{W}$.

The reader should also verify that when w is at 0 and at h the centre of gravity of the system is in its highest position, whereas at *h′* it is at its lowest.

General Considerations on Conservation. When work is done against gravity it confers *energy* upon the body in amount equal to the work done. Thus the power of doing work is not changed in amount during the process; or, as we have said (p. 120), the total energy is conserved. The question arises whether this is true when work is done against other forces as well as against gravity. Now it can be shown generally that all work against forces whose values depend only upon the relative position of the bodies, *must* have this quality. In the first place, if the body returns to its initial point along the identical path by which it advanced, the force at each stage and the direction it makes with the path are the same, both while going and coming, and the work done while a short length of path is traversed will be equal, but opposite, on the forward and return journeys. Since this is true for each short length of path it must be true for the whole. That is, work being done *against* such a force on the forward journey—the kinetic energy decreasing, of course, at an equal rate—the body is thereby placed in such a position that work can in turn be done by the force; the consequence of this work done is that the body regains the whole of its kinetic energy, if it returns to its initial position.

But this is true whatever the shape of the path may be, provided it is a closed curve. For, consider the force of any particle on another; assume that its value depends only on the distance apart of the particles and that it is directed along the line joining them. Let O be one of them considered as fixed, and A the other taken as movable. Let A move along any line AB and let $\overline{PQ}$ be a length of the path so short that the force (whose direction joins the particle with O) may be taken as uniform along it. The work done is F . $\overline{PQ}$ cos MPQ; this equals F . $\overline{pq}$, p and q being points on the radius $\overline{OB}$ produced, and on the same circles as P and Q respectively; and this is equal to the work that would be done along the path $\overline{pq}$ (Fig. 104). Since each elementary path may be treated in the same way, it follows that the work done during the whole path $\overline{AB}$, whatever the shape may be, is equal to the work done along the path $\overline{aB}$, a and A being at the same distance from O. This result is true independently of the manner in which the body O, acting as a centre of force, is moving, it being understood that the path $\overline{AB}$ is determined relatively to O.

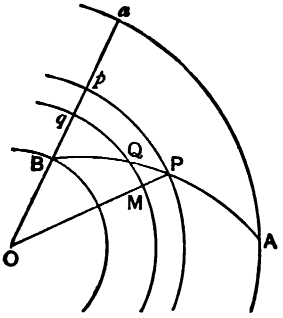


Fig. 104.

It follows from this theorem that, whatever may be the law of variation of the force with distance, the total work done in going completely round a *closed* path is zero. This may be otherwise expressed by saying that a definite amount of work having been done against the force in going from A to B along any path, the body is thereby placed in such a position that an *equal* amount can be done *by* the force during the return journey along the same, or any other, path to the initial point A. Work done against such a central force confers upon a body the power of doing work in equal amount, and we are therefore justified in regarding the work so done as potential energy, as in the case of gravitation.

This being so, if all bodies consist of particles, the action between any pair consisting of forces of the above kind, then all work done against these forces (at the expense, of course, of some type of energy) will be represented by the appearance of potential energy in equal amount, and the total amount of energy will remain unchanged.

Are all forces in Nature, then, of this type?

At first sight it would certainly seem that they are not. A body is moving on a rough horizontal table; friction, if no other force is acting, gradually brings it to rest. Initially, it has kinetic energy; finally, this has apparently all disappeared. Work has been done against friction at the expense of kinetic energy; but the work so done does not appear as potential energy, for the body does not spontaneously return toward the starting-point. Friction always acts so as to oppose the motion taking place; therefore work is always done *against*, never *by*, it. It would seem, then, that when work is done against friction—and no motion of actual bodies takes place without friction—there is a gradual loss of energy, and consequently that the doctrine of conservation is never really true.

This, however, is only an imperfect view, and though a fuller consideration demands an encroachment on another branch of Physics, *viz.* Heat, a brief outline must be given here. The body which has just done work against friction is *not* in the same state as at first: it is *hotter* than it was. If the quantity of heat to which the rise of temperature is due is measured, it is found to be *proportional* to the work done against friction. Moreover, in all heat engines (such as a steam-engine), work is done at the expense of heat, and again there is proportionality between the two. These correspondences are explained by assuming that heat is itself a form of energy, to which the law of conservation is applicable. This, of course, involves that when work is transformed into heat, or heat into work, the two shall be *equal* to one another, whereas experiment shows only that they are *proportional.* But it must be remembered that the units, in terms of which the two quantities are expressed, are different from one another. The work done in any experiment would be measured in terms of a unit of work (erg, joule, foot-lb.), whereas the heat produced would be expressed in thermal units (calorie, pound-degree). If, however, heat is a form of energy, the calorie is itself a unit of energy, and the factor of proportionality is simply the ratio between two units. Thus, if J ergs of work are transformed into 1 calorie of heat, then the calorie represents J times as much energy as an erg. The factor J is called Joule's equivalent; its numerical value depends upon the units to which it relates: 42,000,000 ergs are nearly equal to one calorie (*i.e.* a gram-degree Centigrade). Again, 1,400 foot-lb. nearly equal one pound-degree Centigrade, and 778 foot-lb. nearly equal one pound-degree Fahrenheit.

Nature of Heat. We have classed other kinds of energy under the head either of kinetic energy or potential energy. Can we place heat in either of these classes, or is it a thing *sui generis?* According to the kinetic theory of matter, the molecules, of which any body is conceived to consist, are in rapid motion, and this motion becomes more intense as the temperature increases; hence, when work is done against friction, energy which would otherwise have appeared as either energy of visible motion, or of the position of the body as a whole, becomes instead either energy of the motion, which is invisible to the eyes, or else energy which arises from the relative positions of the ultimate particles of matter.

In order to realise how a body may possess even a large amount of kinetic energy, although it is not visibly in motion, it must be observed that the total kinetic energy of a group of particles is the sum $\Sigma(\frac{1}{2}mv^2)$, whereas the total momentum is $\Sigma(mv)$. Now, if the particles are moving not all in the same direction, it is possible for the latter to be zero; but the former cannot be zero *unless* EACH *of the particles is at rest;* for the square of a quantity is positive whether the quantity is positive or negative.

The supposition is, then, that heat consists, in part, of vibratory motions of the molecules, these motions being of a random character; so that, in any small portion of the body containing a large number of the molecules, the motion of any molecule is as likely to be in any one direction as in any other, and the total momentum is zero. But the whole body is, nevertheless, a store of kinetic energy; and if we only knew how to bring *every one* of these molecules to rest, we could cause the body to do work equal to the kinetic energy it possesses. Unfortunately we are not able to do this completely, though the reverse process is always possible. The energy of a falling stone is wholly converted into heat when it strikes the ground, but the heat produced could only in part be reconverted into visible energy. The essential difference between the direct and reverse transformations is very similar to that between tangling up a skein of silk and then disentangling it. Again, it is similar to the ease with which a confused mass is produced as the result of the impact of two trains, while the original state of things can only be restored by dealing *individually* with each portion of the wreck. A closer analogy is presented if we compare the molecules to a large number of horses scampering in all directions over a plain. Their energy may be immense, but it is unavailable. Harnessed to a carriage, they would only tear it to pieces. If a second field of horses, moving less energetically, were also harnessed to it, the carriage would be drawn in the direction of the stronger, and some of the energy would be transformed into work. But to transform all the energy of their motion into useful work, it would be necessary to deal with each horse individually—to harness it when it happened to be going in your direction, and to replace each by another when it had been brought to rest.

The conditions under which *part*—and part only—of the random motion constituting heat is transformed in an engine into regular or progressive motion of the train cannot be considered here. What we wish to emphasise in this place is that, according to the kinetic theory, a satisfactory account can be given of the nature of heat. If, however, it is desired to state the doctrine of the conservation of energy, without introducing any hypothesis as to the nature of heat, it can be expressed by saying that the amount of heat ultimately obtainable from a definite amount of energy is a fixed quantity, in whatever way the transformation takes place.

Perpetual Motion. The doctrine of the conservation of energy is closely connected with the denial of the possibility of what is known as "perpetual motion." All searchers after "perpetual motion" have had in view the expectation of obtaining work from a machine without any expenditure of energy. A mill wheel that will turn without water, a train that will go without fuel, or (in this electrical age) a battery that will light your workshop on alternate days, and at the same time charge up a similar battery for use on the intermediate days,—all these are typical cases of what it is sought to obtain.

Now it is evident that this is "perpetual motion," and something more. The doctrine of conservation does not by itself assert the impossibility of a falling stone rebounding from a plane to the same height as that from which it fell, again falling and rising, and so on perpetually. But taken along with the experimental fact that no motion occurs unopposed by friction, it does deny the possibility. And when the problem is, not merely to arrange that the same motion shall be automatically repeated, but to contrive that useful work shall at the same time be performed, it asserts that, not only must you introduce as much energy into your mechanism as you wish to be transformed into such useful work, but you must introduce *more*, *viz.* as much more as will be inevitably frittered away uselessly into heat.

Loss of Energy during Collision. When two balls collide we have seen how to obtain their velocities afterwards in terms of their initial velocities and the coefficient of restitution : from these data their loss of kinetic energy can be calculated. We will first consider a case in which the bodies are inelastic. Then we may write

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v.$$

The kinetic energy just before collision is

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2,$$

whereas just afterwards it is

$$\tfrac{1}{2}(m_1 + m_2)v^2,$$

which by the first equation

$$= \frac{(m_1u_1 + m_2u_2)^2}{2(m_1 + m_2)}.$$

The *decrease* of kinetic energy is therefore

$$\tfrac{1}{2}\left[m_1u_1^2 + m_2u_2^2 - \frac{(m_1u_1 + m_2u_2)^2}{(m_1 + m_2)}\right],$$

and this equals

$$\tfrac{1}{2}\left(\frac{m_1m_2u_2^2 + m_1m_2u_1^2 - 2m_1m_2u_1u_2}{m_1 + m_2}\right)$$

$$= \tfrac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_2 - u_1)^2.$$

Since this is proportional to the square of the initial relative velocity it is always positive; the kinetic energy, in other words, is always reduced by the collision. The balance is converted into heat in the bodies as a result of the work done against frictional forces during the deformation which the bodies undergo. Whether we regard this portion as wasted or not depends upon what we wish to effect. If the collision is between a hammer and a nail, the desire is to drive in the nail; and since in penetrating the board into which it is being driven it must do work against the resistance to penetration, the amount it can so do at each stroke will be greater (for a given initial energy) if little is converted into heat at each blow. The ratio which the amount of energy utilisable bears to that supplied may be called the efficiency of the blow.

The ratio in the case of the hammer and nail (the former being the body of mass, m_1), if the nail is initially at rest, is

$$\frac{1}{2}\frac{m_1^2u_1^2}{m_1 + m_2} \div \frac{1}{2}m_1u_1^2,$$

and this ratio is $\frac{m_1}{m_1 + m_2}$. This is great if m_1 is great compared with m_2; that is, a hammer is more efficient the more massive it is compared to the nail.

On the other hand, in flattening a rivet the desired aim is to produce as much deformation as possible. It is the energy wasted in the previous case which is utilised in this. Again, taking the rivet as initially at rest, we require that

$$\frac{1}{2}\frac{m_1m_2}{m_1+m_2}u_1^2 \div \frac{1}{2}m_1u_1^2$$

shall be large. This ratio is $m_2/(m_1+m_2)$, and is large if m_2 is large compared with m_1. Thus the hammer should be *light*, compared with the rivet, in order to secure efficiency. The only useful way of satisfying the requirement is to make the rivet effectively massive by putting it in contact with a massive anvil, so that one cannot move without the other. The anvil is then part of the mass, m_2, and considerable efficiency can be obtained.

These expressions assume that no rebound takes place. By inserting the values found on p. 63 for v_1 and v_2, where the coefficient of restitution is e, into the expression

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2,$$

we obtain

$$\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_2-u_1)^2(1-e^2)$$

as the value of the energy converted into heat. If e were unity the amount would be zero, but this value never occurs in reality.

EXAMPLES TO CHAPTER XV.

1. Find the work done against gravity in excavating a cylindrical shaft, 120 ft. deep and

4 ft. diameter, the mean specific gravity of the material being 2·6.

2. Find the work done in tilting a ladder into the vertical, its mass being 30 lb. and its centre of gravity 10 ft. from the lower end.

3. Find the average work done per ton against gravity in building a uniform wall 20 ft. high.

4. Find the work done against gravity in rolling a barrel of 100 lb. wt. 10 ft. along a plane inclined at $\sin^{-1}(\cdot 1)$ to the horizontal.

5. Find the horse-power of an engine that will take a train weighing 100 tons up an incline of 1 in 100 at the rate of 20 miles per hour, the resistance arising from friction amounting to 20 lb. wt. per ton.

6. A bicycle can just move down a slope of 1 in 100 with constant velocity of 8 miles per hour without the rider working. At what rate must the rider do work in order to take it up a slope of 1 in 200 with the same velocity, assuming the frictional resistances to be the same? Mass of machine and rider, 200 lb.

7. Explain in a general way how the wheels of a cart, moving with constant velocity, contribute more to the total kinetic energy in proportion to their mass than the rest of the cart does.

8. What fraction of the kinetic energy of a bicycle is possessed by the wheels, assuming that the mass of each is one-fifth the weight of the machine, and that its spin-radius is three-quarters the radius of the wheel?

9. Calculate the kinetic energy of a loaded cart travelling 4 miles an hour, the total mass being 16 cwt., and that of each of the two wheels being 56 lb. Take the radius of each wheel as 2 ft. and its spin-radius $1\frac{1}{2}$ ft.

10. Apply the principle of energy to calculate the velocity gained by the weights in an Atwood's machine while moving a given distance from rest, allowing for the rotation of the pulley.

11. Two masses of 6 and 5 lb. hang over a smooth peg. After moving for 2 sec. the 5 lb. mass catches up another mass of 1 lb. previously at rest. Determine the subsequent motion.

[*Hint.* The total momentum is the same just after as just before the collision.]

12. Find the kinetic energy which disappears when the collision takes place.

13. A 2 lb. hammer strikes a 1 oz. nail with a velocity of 2 ft. per sec. Find the kinetic energy just before and just after the hammer strikes, assuming hammer and nail to be inelastic. If the nail penetrates a quarter of an inch at each blow, find the average force opposing penetration.

14. In the former example calculate the fraction of the initial energy which is used in driving the nail.

15. Show that a hammer should be more massive than the nail it drives, and less massive than the rivet it fixes. Hence explain the use of a massive anvil.

16. A hammer drives a nail into a board which

is perfectly free to move. Show that each blow is less effective than if the board is fixed.

17. The nail in problem 13 is driven into a perfectly free board of mass 10 lb. Find the total kinetic energy which is spent in driving the nail.

18. A pile, whose mass is 1 cwt., is driven 1 ft. into the ground, at each blow of a hammer (mass 3 cwt.) falling 30 ft. What force, in addition to its weight, would cause the pile to sink slowly into the ground. (Assume everything inelastic.)

CHAPTER XVI

THE PRINCIPLE OF VIRTUAL WORK

We proved (page 113) that the work done by a system of forces acting on a particle is equal to the work that would be done by their resultant. Now, when a particle is in equilibrium the resultant force is zero. It follows, therefore, that if we imagine the particle to undergo any displacement, *the forces being supposed to remain unchanged in direction and magnitude during the displacement*, the work done by them will be zero. In any displacement that might *actually* take place, the forces will in general change in direction or in magnitude or in both respects. To distinguish a displacement taking place under the conditions laid down from one occurring under actual conditions, it is called a *virtual displacement*, and the work done during it is called *virtual work*.

The above statement may be otherwise expressed in the following way:

The virtual work done in a virtual displacement of a particle from a position of equilibrium is equal to zero.

The mode of applying this theorem to determine the position of equilibrium will be clear from an example:

Take the case of a simple pendulum. Assume that the equilibrium position is as shown in Fig. 105. Let a virtual displacement, s, take place vertically; the work done would be, if θ defines the equilibrium position, $T \cos \theta . s - W . s$, which may be equated to zero. Again, if we imagine a horizontal virtual displacement,

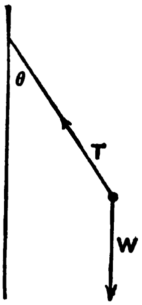


Fig. 105.

$$T \sin \theta . s + W \times \text{zero} \times s = 0.$$

The latter equation gives either $T =$ zero, or $\theta = 0$; a zero value of T is inconsistent with the first equation, hence θ must equal zero, and consequently $T = W$.

If we divide each of the two equations by s, they become those that we obtain by equating the sum of the vertical and also the sum of the horizontal components of force separately to zero. The latter is the more direct way in this case, for the principle of virtual work only introduces a superfluous s into the equations.

To illustrate how the principle may give special information even when applied to a particle, let a displacement take place *perpendicularly to* T, *i.e.* parallel to the tangent of the arc of possible real displacement near the equilibrium position. The work done by T in this displacement is zero; and since the only other force is W, the work done by W must therefore also be zero. Hence the tangent to the arc of possible motion near the equilibrium position must be *horizontal.*

Application to a Rigid Body. If a rigid body is in equilibrium, each particle of it also is; consequently the virtual work done by the forces in a virtual displacement is zero. If we imagine a rigid body to undergo a *translation*, the forces remaining unchanged in direction and magnitude throughout, the work done by the *internal* forces is necessarily zero. For, if we consider the equal and opposite forces between any two particles of the body, and remember that the displacement of both particles is the same when a translation takes place, it is clear that the amounts of work done by the two forces are equal and opposite, and cancel each other, therefore, when added together. Since this is true for the quantities of work done by the forces between any pair, it is true also for the sum. Hence the amounts of work done by the *external* forces must also be zero.

Again, the application of this theorem merely gives the same equation as is derived by equating to zero the sum of the components of the external forces in any direction.

The following theorem is of greater importance:

If *any small* virtual displacement consistent with the rigidity of a body be made about its equilibrium position, the sum of the works done by the external forces is zero.

For considering any particle, the principle of virtual work may be applied to it. Consider now the work done by the internal forces between any pair of particles.

Let their equilibrium positions be A,B, and their displaced positions be A′,B′; then $\overline{A'B'} = \overline{AB}$, for in a rigid body the particles retain their relative positions. Let the mutual force be F, as shown. Then the work done in the displacement if the forces remain unchanged is $F \times \overline{Am} - F \times \overline{Bn}$, where $A'm$ and $B'n$ are perpendicular to $\overline{AB}$. Now $\overline{Am} - \overline{Bn} = \overline{A'B'}(1 - \cos\theta)$, and *if the angle* θ *is small* this is nearly $\overline{A'B'} \cdot \theta^2/2$. Now, if θ is taken small enough, this is indefinitely small compared with $\overline{BB'}$, which is, at least (*i.e.* when A′ and A coincide), proportional to θ. Hence, if an external force at B has a finite component in the direction $\overline{BB'}$, the work done by it when θ is made infinitely small will be infinitely great, compared with that done by the internal forces, F. Since this applies to all the particles, we may say that the work done by the internal forces in the rigid displacement considered is indefinitely small in general, compared with the work done by the

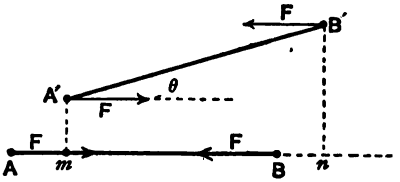


Fig. 106.

external forces. Since the total work done is zero, if the displacement is about the equilibrium position, that done by the external forces must also be sensibly zero.

It must be carefully noted in applying this theorem that the displacement must be small, that the forces acting at various parts of the body are to be supposed acting throughout the motion at the same points in the body, and to be unchanged in direction and magnitude, and that the displacement must be such as leaves the size and shape of the body absolutely unchanged; in other words, it must be such as the rigid body could actually make.

Example. Take the case treated on page 171. Let the small displacement selected be such that the rod slides on the smooth wall and over the smooth peg. Since each of the points at which R_1 and R_2 act moves at right angles to the forces thereat, the work done by each is zero. The only other external force is the weight of the rod acting vertically downwards through the centre of gravity. Since the total work is to be zero if the position is that for the equilibrium of the rod, the work done by gravity must also be zero; that is, the centre of gravity must move horizontally during the displacement. Now, if we sketch in the path which the centre of gravity traverses, as the end of the rod is moved into all possible positions on the wall while resting throughout on the peg (Fig. 107, dotted line), the points of the path which are horizontal satisfy this criterion. One of these corresponds to the case where R_1 points outwards from the wall, the other to the case which might arise if the wall could exert a *pull* (instead of a push) on the rod, and therefore R_1 pointed inwards. To decide which is which, some other virtual displacement—such as a translation—must be made, and the theorem again applied.

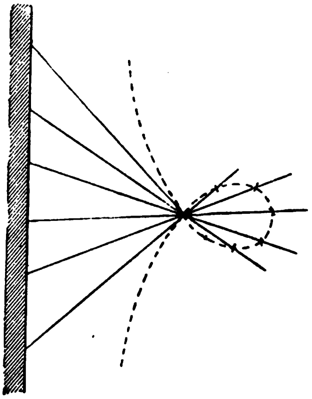


Fig. 107.

Application to Non-rigid Bodies. If the body is not rigid, or if the virtual displacement made is not consistent with the rigidity of a body, work will be done by the *internal* forces, and it is only if this work be included that the total work is zero for motion in the neighbourhood of an equilibrium position.

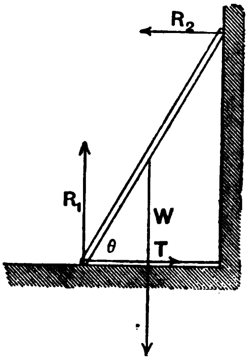


Fig. 108.

As an example, take the case of a uniform beam on a smooth ground, and leaning against a smooth wall, to which its lower end is connected by a string. Then, if the virtual displacement be that in which the beam slides

against wall and ground, work will be done, not only by W, but by the tension in the string, for the string would require to lengthen during the motion; R_1 and R_2 however, do no work. Let θ_1 be the inclination of the rod in its equilibrium position, and $2l$ its length; in the displacement θ_1 changes to θ_2. The initial height of the centre of gravity is $l \sin \theta_1$ and the final height $l \sin \theta_2$. The initial length of the string is $2l \cos \theta_1$ and the final length is $2l \cos \theta_2$.
Hence

$$Wl(\sin \theta_1 - \sin \theta_2) = T\,2l\,(\cos \theta_2 - \cos \theta_1)$$

when $\theta_2 - \theta_1$ is indefinitely small.

Since the change in θ is small, the difference of sines is nearly $\cos \theta_1 \times (\theta_1 - \theta_2)$, and the difference in cosines is nearly $\sin \theta_1 \times (\theta_1 - \theta_2)$; therefore, dividing both sides by $l(\theta_1 - \theta_2)$, we obtain

$$W = 2T\frac{\sin \theta_1}{\cos \theta_1} = 2T \tan \theta_1,$$

the same relation as was found on p. 174 by another method.

If R_1 or R_2 is required, a displacement must be made in which work is done against one or other of these forces. For example, work is done against R_2 in a horizontal displacement from left to right; and such a displacement equal to s gives

$$Ts - R_2 s = 0,$$

$$\text{or} \quad T = R_2.$$

Stability of Equilibrium. If a real displacement from an equilibrium position be given to a body, either positive or negative work is in general done during it, because the forces do not retain their equilibrium value. We shall only consider cases for which gravity is the only force which does work. If this work is positive it corresponds to a decrease of potential energy, and consequently an increase in kinetic energy; hence, any velocity with which the body is started from its equilibrium position will be increased during the displacement, and the body will therefore tend to move further from the position of equilibrium.

If, however, the potential energy *increases*, the kinetic energy must *decrease*; hence the velocity given to the body initially by the force which started it from the equilibrium position will be taken from it; in other words, gravity opposes the motion from the initial position. In the former case the equilibrium is called *unstable*, and in the latter *stable*. Now, if the body is in its lowest possible position, the potential energy can only increase during any small displacement; on the other hand, if it is in its highest possible position, it can only decrease. Thus the criterion as to whether equilibrium is stable or not is that for stable equilibrium the potential energy must be a minimum; or, what comes to the same thing, the centre of gravity of the body must be as low as possible.

If the change of potential energy is zero, the equilibrium is said to be *neutral.* Any initial velocity will remain constant throughout the displacement. Each position throughout the displacement is a position of equilibrium.

Thus, if a homogeneous sphere moves on a smooth horizontal table, no work is done by the table, and gravity does no work, however large the displacement, since the centre of gravity remains at a constant height above the table. Every position is an equilibrium one. A hemisphere is in stable equilibrium when its base is horizontal. A top at rest is in unstable equilibrium when balanced on its peg.

These considerations only apply to the case of rest, and it must not be inferred that the centre of gravity of a body always falls as low as possible. For example, when a top is spinning the centre of gravity mounts as high as possible: and if displaced from this highest position will return there.

Examples to Chapter XVI.

1. Work out the examples to Chapter XI. page 180, by the method of virtual work.

2. In a parallelogram formed by four rods with smooth joints, elastic strings are stretched along the diagonals. Determine the ratio of the tensions in the strings in terms of the lengths of the diagonals.

3. Show that a homogeneous hemisphere resting on a horizontal table is in stable equilibrium.

4. A cylinder of the same radius is mounted on the top of the hemisphere in the previous question. What is the height that the cylinder must have in order that the equilibrium may become unstable?

CHAPTER XVII

ELEMENTARY MACHINES

THE object of any mechanism is to change the directions, positions, or magnitudes of forces, or of motions, so as to make them serve some particular purpose. For example, a body can be drawn up an inclined plane with less force than that required to lift it vertically. In this case the object is to enable a more massive body to be raised than would be possible without the use of the plane.

By the use of a simple pulley wheel a body may be raised by applying a *downward* force; it is a consideration of convenience of direction which makes such a wheel useful. We shall see that by combinations of pulleys the force required may be reduced as in the case of an inclined plane. Again, on a cycle, a man propels himself forward by the downward, or even backward, application of a force.

As an illustration of those mechanisms in which the object is merely to change the motion the case of the gearing of a clock may be cited. Here by means of suitable gearing one hand is made to go round twelve times as rapidly as the other.

The student should for himself examine all the mechanisms with which he is familiar, and see that in all of them the one or other of the aims stated at the beginning of this chapter is realised.

In a complicated mechanism—such as a loom—hundreds of different operations may be simultaneously performed, by means of a *train* of mechanism; that is a series of parts of more elementary character linked together with one another. Many of these parts are of very frequent occurrence, and are called Elementary machines. We shall briefly discuss the more common of them in the present chapter. All the dynamic ones have the characteristic that force applied at one part enables work to be done against a force. The applied force is called the *Effort* [very commonly it is called the *Power*, but the use of this term should be discouraged as it has another and very important denotation]; the force against which work is done is called the *Resistance*; the ratio of the Resistance to the Effort is taken as a measure of the *Mechanical Advantage* (M.A.). If the forces are equal to one another the M.A. is *unity*, which requires to be interpreted as meaning that, as far as magnitude is concerned, there is no advantage at all gained. If the ratio is less than unity, a greater force is exerted than would be necessary without the mechanism; there is then a *Disadvantage*. But these phrases must not be interpreted too literally; for in the multifarious cases which arise, the convenience gained in other respects may more than compensate for what is technically a mechanical disadvantage.

In considering mechanisms it is customary—unless otherwise stated—to deal only with the case of equilibrium; in other words, the machine is assumed to be working steadily. While the machine is starting (or stopping) the relations between the forces will have values different from those so calculated. During starting the Effort will in all cases require to be greater in order to produce the acceleration of the various parts of the mechanism. During steady motion the work done by the Effort must be equal (apart from friction) to the work done against the Resistance. The distance passed over in a given time by the points of application of these two forces can be calculated from the configuration of the mechanism, and since the quantities of work are the same, must be inversely as the forces. This fact enables the mechanical advantage to be easily calculated when friction can be taken as being negligible. In every case the Effort will be denoted by E and the Resistance by R.

(*a*) **Rope or Chain.** Though not usually classed as a Simple Machine, a rope is in reality the simplest of all. It enables a body to be moved from a distant point. The M.A. is unity. Connecting cords, chains, or wires in machinery are of frequent occurrence, and play a similar part.

When a body is to be raised to a great height a disadvantage arises from the weight of the rope itself.

(*b*) **Inclined Surfaces.**

1. *Inclined Plane.* This case has been already considered (Chapters VI. and XII.), except that the M.A. has not been explicitly stated. The results are here enumerated. The resistance, R, is in each case the weight of the body.

α. Effort parallel to smooth plane—

$$\text{M.A.} = \frac{\text{R}}{\text{E}} = \frac{1}{\sin\theta} = \frac{\text{length}}{\text{height}}.$$

Effort parallel to rough plane—

$$\frac{\text{R}}{\text{E}} = \frac{1}{\sin\theta + \mu\cos\theta} = \frac{\cos\alpha}{\sin(\theta + \alpha)},$$

where α = limiting angle of friction.

β. Effort horizontal and plane smooth—

$$\frac{\text{R}}{\text{E}} = \frac{1}{\tan\theta} = \frac{\text{base}}{\text{height}}.$$

Effort horizontal and plane rough—

$$\frac{\text{R}}{\text{E}} = \frac{1}{\tan(\theta+\alpha)}.$$

γ. Effort at best angle of traction (plane rough)—

$$\frac{\text{R}}{\text{E}} = \frac{1}{\sin(\theta + \alpha)}.$$

2. *The Screw.* A screw consists, in reality, of an inclined plane *wound round a cylinder.* As the screw turns through one revolution it rises through a height equal to the distance from one spire to the next. This distance is called the *pitch* (p). If the effort is applied at the circumference of a wheel of radius b, or a rod of length b, the work done during a revolution is $E \,.\, 2\pi b$; the corresponding work done against the vertical resistance is Rp; and since these quantities of work are equal the

$$\text{M.A.} = \frac{R}{E} = \frac{2\pi b}{p}.$$

This assumes that the surfaces are smooth. If they are rough, work is also done against friction. The force of friction is $\mu R \cos \theta$ (as in an inclined plane); this is nearly equal to μR, for θ is always small. The work done against friction is $2\pi r \,.\, \mu R$ where r is the radius of the screw. Consequently

$$2\pi b \,.\, E = Rp + 2\pi r\mu R,$$

and the mechanical advantage is

$$\frac{R}{E} = \frac{2\pi b}{p + 2\pi r\mu}.$$

(3) *The Wedge.* The principle of the wedge is employed in an ordinary *axe* and in a *chisel.* The sharp edge is inserted in the substance to be split, and driven in by blows in the direction E; the pressures R on the sides constitute the resistance against which useful work is done,

When the wedge has been completely driven in, the points of application of the resistances have separated by an amount equal to the breadth of the wedge. Consequently, if the wedge is smooth

$$\text{M.A.} = \frac{\text{R}}{\text{E}} = \frac{\text{height}}{\text{breadth}}.$$

For a rough wedge, the corresponding work done against friction is nearly $2\mu\text{R}h$, frictional work being done on both sides of it.

$$\therefore\ \text{M.A.} = \frac{h}{b + 2\mu h}.$$

An ordinary knife is used as a wedge when it is simply pressed down on a body as in cutting cheese. When it is drawn to and fro, its effective angle is diminished; moreover, it is then used as a *saw*, which consists of a long row of wedges, which can be forced into the body by a motion inclined to the freshly formed surface; a longitudinal motion then tears the wedges free by crumbling away the wood or metal caught between them.

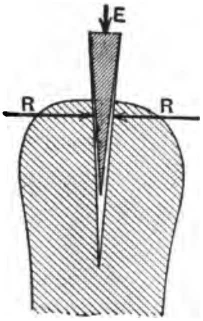


Fig. 109.

A *measuring wedge* is an example of the use of a mechanism for magnifying a motion. The scale is marked on the base, and the wedge, being of small angle, a very small *height* corresponds to a considerable length of base. If the wedge be inserted in a narrow opening, the width of the opening can be easily read off on the extended scale.

Levers. 1. A lever consists of a rigid bar with one point kept fixed so that a rotary motion may be given to the bar. An effort applied at any point of the lever can do work against a resistance located at any other point. If the distances of these two points from the fixed axle (known as the *fulcrum*) are respectively a and b the amounts of work done by the two forces during a small rotation are $2\pi a\mathrm{E}$ and $2\pi b\mathrm{R}$; the mechanical advantage is therefore

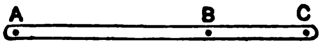


Fig. 109A.

$$\frac{\mathrm{R}}{\mathrm{E}}=\frac{a}{b}.$$

Several cases may arise according to the relative positions of the forces; all are included in the above formula. Take points A and C at the two ends, and let B be *any* intermediate point; when R and E are at A and C and the fulcrum at B it is a lever of the *first* kind; with the fulcrum at A, R at B, and E at C, it is of the *second* kind; with the fulcrum at A, R at C, and E at B, it is of the *third* kind. In the second kind a is necessarily greater than b, in the *third* it is necessarily less, and in the first it may either be greater or less: the mechanical advantage varies accordingly. A *crowbar* is a lever of the first kind; a pair of household scissors consists of two such levers, each of which is a wedge, joined together. A pair of nut-crackers is a *double* lever of the second kind; the force crushing the nut is necessarily greater than that exerted by the hand. In the human fore-arm the elbow is the fulcrum; the resistance is applied at the hand, while the effort is in the tendon attached between; it is therefore a lever of the third kind, and the force in the tendon is necessarily greater than the weight lifted thereby. A pair of spring shears, such as are used for clipping sheep or hedges, is a double lever of the third kind.

An oar is a lever of the second kind, the blade being the fulcrum; the force on the rowlock is therefore greater than that applied at the handle. In calculating the force driving the boat forward (against the resistance of the water) it must be borne in mind that, besides the forward force, R, at the rowlock, there is a backward force on the seat and stretcher equal to E (to which it is the reaction). Thus the effective force on the boat is R – E. Since, in the general notation $R/E = a/b$

$$\text{Force propelling boat} = R - E = \frac{a-b}{b}E.$$

This is also the relation that would have been obtained if the boat (and rowlocks) were regarded as fixed, and each rowlock as a fulcrum: E being at the handle; and R, the resistance, at the blade *pushing the water backwards* with respect to the *stationary boat.* According to this interpretation the oar would be a lever of the first kind. However instructive this alternative interpretation is in connection with the relativity of motion, the former interpretation which regards the boat as the moving object is more in accordance with ordinary convention.

We have tacitly supposed the forces to act normally to the length of the bar; if they do not, then *a* and *b* are distances measured perpendicularly to the forces.

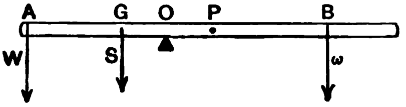


Fig. 110.

2. *The common steelyard.* This is essentially a lever, the object of which is in this case merely to compare two forces. The difference in the calculation from the above cases is that the weight of S the steelyard itself is not negligible. The position B at which a weight, ω, will *balance* (*i.e.* produce equilibrium along with) S at the centre of gravity G, and W at A, when the knife edge is at O, can, of course, be calculated from the principle of work; for variety, we will employ the principle of moments. This gives

$$\omega . \overline{OB} = W . \overline{AO} + S . \overline{GO},$$

if G is on the same side of O as A. Let a point P be chosen between O and B, such that

$$S . \overline{GO} = \omega . \overline{OP}$$

$$\text{then } \omega . \overline{PB} = W . \overline{AO}.$$

Hence equal increments of distance of B from P correspond to equal increments of values of W which a given weight ω will counterpoise. If the arm is to be graduated, P should be marked *zero*, and a length $\overline{\text{PB}}$ equal to $\overline{\text{AO}}/\omega$ will determine the mark corresponding to unit increment in W. If the centre of gravity is on the same side as ω, then P must be taken on the opposite side, but is determined by the same rule.

3. The *ordinary balance* is also a mechanism by which one weight is balanced against a second. The modern student is familiar with the general appearance of the balance; this need not therefore be elaborately described; we shall denote the point of suspension by O, and the knife edges, at which the scale pans are hung, by A and B. We shall first show that the scale pans and any weights in them may be supposed concentrated at A and B respectively. For, the knife edges are free to turn on the planes whereon they rest, and whatever the position of the beam the resultant of the weights of a scale pan together with the weights in it is a vertical force passing through the knife edge. In fact, A is the *centre* of this group of parallel forces. Similarly for B and the scale pan hung at it. Let G be the centre of gravity of the beam of the balance, together with the *empty* scale pans supposed concentrated at A and B. In a properly constructed balance the beam is intended to be horizontal when the pans are unloaded; in

this case, since G is vertically below O when the beam is free to move the line OG must be perpendicular to AB. When weights are placed in the pans the line AB when permanently at rest will in general deviate from horizontality and the deviated position is shown in Fig. 111.

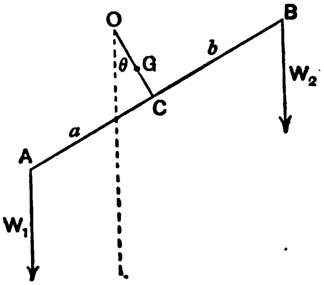


Fig. 111.

This position can be calculated by equating the sum of the moments about O to zero. Let AC = a, CB = b (these are the *arms* of the balance), OG = h, OC = k, θ = deviation, W_1 and W_2, the weights *in* the pans, and B the weight of the beam and empty pans.

Then

$$W_1 (a \cos\theta - k \sin\theta)$$
$$= W_2 (b \cos\theta + k \sin\theta) + Bh \sin\theta$$

whence

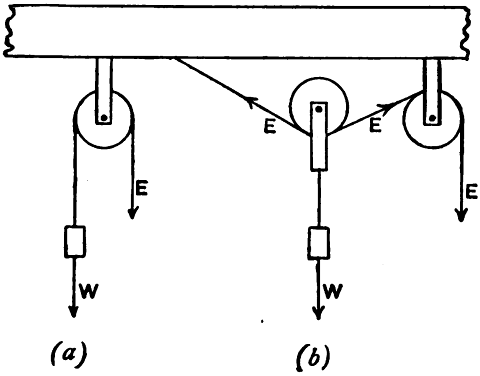
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{W_1 a - W_2 b}{Bh + (W_1 + W_2)k}.$$

Consider first the case in which $b = a$; the numerator becomes $(W_1 - W_2)b$. Since a balance is used for comparing weights, it is essential for accuracy that a small difference in weights shall be easily detected. If the weights are equal, $\tan\theta = 0$, and the beam balances in the same position as when the pans are empty; a small difference in the weights will produce a deviation and the greater this is the more easily will it be detected. The ratio $\tan\theta / (W_1 - W_2)$ is $a / \{Bh + (W_1 + W_2)k\}$ which increases as a increases, and diminishes as h, k, and $(W_1 + W_2)$ increase. Thus a balance is less *sensitive* when heavily loaded and more sensitive when the arms are long, and also when h and k are small. There is a serious objection, though, to pushing the sensitiveness as far as possible. Rapidity of motion is necessary if a large number of weighings are required; and a discussion of the motion as in the case on p. 294 shows that the time period increases as a increases and diminishes as h, k, and W increase. That is to say, the same circumstances which correspond to greater sensitiveness also correspond to increased slowness of working. The value of a has the most proportionate influence in prolonging the period; and consequently, in modern balances, the length of arm is made short, and the requisite sensitiveness is secured by diminishing h and k.

When the arms are not equal, the beam becomes horizontal when $W_1a = W_2b$, *i.e.* the weights that balance are inversely proportional to the adjacent arms. To avoid knowing the ratio of the arms, the weights may be interchanged. The geometric mean of the apparent weights of a body in the two cases is its true weight.

Pulleys. A simple pulley is a single grooved wheel with a rope round it. Apart from considerations of friction, a peg or cylinder incapable of rotation would answer equally well. It may be used either to change the direction of a force, without affecting its magnitude, as in Fig. 112*a*, or it may vary both, as in Fig. 112*b*.

In the latter case, if 2θ is the angle between the inclined ropes, and w is the weight of the pulley,



(*a*) (*b*)

Fig. 112.

$$W + w = 2E \cos \theta, \text{ and}$$

$$\text{M.A.} = 2 \cos \theta - \frac{w}{E}.$$

Pulleys are generally used in combination with one another. A second pulley is shown in Fig. 112*b*, changing the direction alone of the effort. The following combinations are of frequent occurrence.

First System. The pulleys are gathered together into two blocks, all in one block having a common axis. A single cord is passed round the two blocks, as shown in Fig. 113, where it is attached to the upper block; or it may start at the lower block instead. In either case, when W rises through any distance b, each of the lengths of cord shortens by an equal amount, and consequently the effort E moves through a distance nb, where n is the number of lengths between the two blocks; hence the M.A. is n, neglecting the weight of the lower block. If this weight, w, be taken account of, then

$$W + w = nE, \text{ and the M.A. is } n - \frac{w}{E}.$$

This result can also be determined by considering the equilibrium of the lower block. Since the tension in each part of the cord is the same (neglecting the weight of the cord), *viz.* E, the upward force on the lower block is nE, the downward force is $W + w$; and these must be equal.

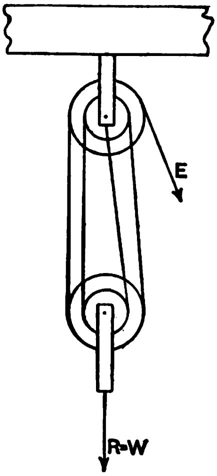


Fig. 113.

In reality the ropes are stiff and the pulleys are not smoothly mounted. The effect arising from these causes is a diminution of the tension in the rope as it passes over a pulley; we can allow for it approximately by supposing that this diminution is always in a fixed ratio, k (say), where k is a proper fraction. The equation of equilibrium then becomes

$$E\,(k + k^2 + k^3 + k^4 + \ldots + k^n) = W + w$$

$$\text{or} \qquad \frac{E\,(1 - k^n)}{1 - k}\,k = W + w$$

$$\text{Whence} \qquad \text{M.A.} = \frac{W}{E} = \frac{(1 - k^n)k}{1 - k} - \frac{w}{E}.$$

When the number of pulleys is infinite this becomes

$$\frac{k}{1 - k} - \frac{w}{E}.$$

This represents a limiting value of the advantage derivable from the employment of actual pulleys.

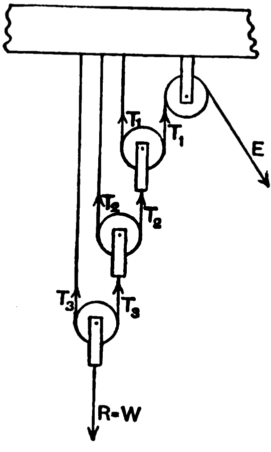


Fig. 114.

If k were as small as one-half, there would be a mechanical disadvantage.

Second System. The pulleys are arranged as shown (Fig. 114). One end of each cord is attached to a fixed support, the other is attached to the next pulley frame, except in the case of the last, for which it passes over a fixed pulley, and the effort is applied at its free end.

Let w be the weight of each pulley, and let the tensions in the cords be as shown Then

$$W + w = 2T_3$$
$$T_3 + w = 2T_2$$
$$T_2 + w = 2T_1 = 2E.$$

Hence $W + w = 2^2T_2 - 2w$
$= 2^3E - 2^2w - 2w,$

or if there are n movable pulleys

$$W = 2^nE - (2^{n-1} + 2^{n-2} + \ldots + 2 + 1)w.$$

To find the sum, S, of the quantity in brackets, multiply it by 2 and subtract from S itself.

$$2S - S = 2^n + 2^{n-1} + \ldots + 2^2 + 2 - (2^{n-1} + 2^{n-2} \ldots 2 + 1),$$

or $S = 2^n - 1.$

Hence $W = 2^nE - (2^n - 1)w,$

and the M.A. is $2^n - \dfrac{2^n - 1}{2^n}\dfrac{w}{E}.$

When n is large this is nearly $2^n - \dfrac{w}{E}.$

If we allow for stiffness and friction as before the equations become

$$\begin{aligned} W + w &= (1 + k)T_3 \\ T_3 + w &= (1 + k)T_2 \\ T_2 + w &= (1 + k)T_1. \end{aligned}$$

Hence

$$W = (1 + k)^3E - [(1 + k)^2 + (1 + k) + \ldots + 1]w,$$

or for n movable pulleys

$$W = (1 + k)^nE - [(1 + k)^{n-1} + \ldots + 1]w.$$

Let S equal the factor of w;

then $(1 + k)S - S = (1 + k)^n - 1,$

and $W = (1 + k)^nE - \dfrac{(1 + k)^n - 1}{k}w.$

The *Third System* is similar to the previous, turned upside down in appearance. The figure (Fig. 115) shows the case of two movable pulleys.

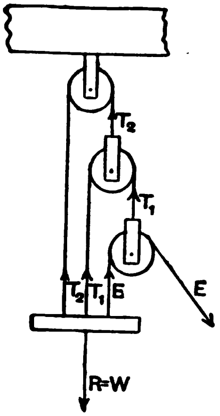


Fig. 115.

The tensions being as shown, we have, each pulley being in equilibrium—

$$T_1 = 2E + w$$
$$T_2 = 2T_1 + w = 2^2E + (2 + 1)w$$
$$T_3 = 2T_2 + w = 2^3E + (2^2 + 2 + 1)w, \text{ etc.}$$

Lastly $W = E + T_1 + T_2 + \ldots\ldots + T_n$ for n movable pulleys.

Or $$W = (2^n + 2^{n-1} + . + 2 + 1)E + [(2^n - 1) + (2^{n-1} - 1) + \ldots 2 - 1)]w$$
$$= (2^{n+1} - 1)\, E + (2^{n+1} - 2 - n)w.$$

Wheel and Axle. This may be called a continuous lever. It consists of two cylinders of different diameters on the same axis. A rope is wound round each, righthandedly in one case and oppositely in the other. When the cylinders are turned the end of one rope rises, and of the other falls. If the radii of the large and small cylinders are a and b respectively, during one revolution one end falls $2\pi a$, the other rises $2\pi b$. Thus, if the effort be applied to the rope that goes round the larger cylinder, and the resistance (a weight for example) be attached to the other

$$2\pi a \,.\, \mathrm{E} = 2\pi b \,.\, \mathrm{R},$$

and the mechanical advantage is a/b as in the case of a lever. Instead of being rotated directly by a rope the axle may be turned by a wheel or a handle, and the effort be applied at the circumference or at the end of the handle; its distance from the axis must then replace a in the formula. The rope may form a continuous loop passing round a movable pulley as shown in Fig. 116.

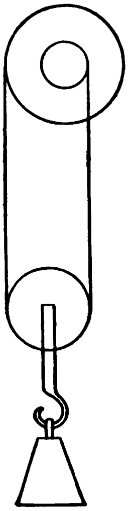


Fig. 116.

When the rope is thick the radii should be measured from the axis to the middle of the rope. When it is *stiff* the M.A. is always less than that calculated from the above formula.

EXAMPLES TO CHAPTER XVII.

1. Calculate the mechanical advantage of a rough inclined plane, of 30° angle and coefficient

of friction $\frac{1}{4}$, for the case (i) of an effort parallel to the plane; (ii) of an effort applied horizontally; (iii) of an effort making an angle of 30° above the plane.

2. How rough must the plane be in each of the three above cases, in order that less force would be required to lift the body without the use of the plane?

3. A wedge weighing 4 lb. is driven 10 in. into a piece of timber by twelve blows of a hammer, of mass 2 lb., which strikes it with a velocity of 8 ft. per sec. If the angle of the wedge is $\frac{1}{10}$ radian, calculate the average resistance to separation of the timber, the coefficient of friction between wedge and timber being $\frac{1}{3}$.

4. With what force must you push down on the above wedge to cause it to slowly split the timber without any blows from a hammer?

5. Find the mechanical advantage of the system of pulleys in which all the movable pulleys are in one block (the rope being attached to the fixed block) when there are four movable pulleys of negligible weight.

6. If the weight of the lower block is 4 lb. wt., and the friction is such that the tension in the rope changes always to $\frac{3}{4}$ its value when the rope passes over a pulley, find the mechanical advantage when the effort is 40 lb. wt.

7. A system of three movable pulleys, each rope being attached to the supporting beam and *to the* previous pulley mount, is used to lift a

a weight of 1 cwt. What effort must be applied (i) neglecting weights of pulleys; (ii) allowing for weight, each being 3 lb.; (iii) allowing for weight of pulleys and also for friction and stiffness of ropes, this being such that ratio of the tensions on the two sides of each pulley is 2/3?

8. To which class of lever does the oar of a boat belong?

9. Give illustrations of the application of the principle of the lever and of the wedge.

10. The time of swing of a balance is 2 sec.; when a weight of 40 grams is attached to the end of the pointer, which is 30 cm. long, the time period becomes 1·5 sec. What is the moment of inertia of the balance?

11. A bar of uniform thickness and density, weighing 2 lb. per ft., is used as a lever of the first kind; the fulcrum is 9 in. from one end of the bar. What must be the length of the bar when its weight balances a weight of 36 lb. suspended at the end of the shorter arm?

12. A weight of 10 lb. is suspended from a single movable pulley, the cords of which are inclined at an angle of 30°. Required the tension in the cord?

13. If the arms of a uniform lever of the first kind be *a* and *b*, and if the weights P and W at the ends be in equilibrium, what is the weight of the lever?

14. If weights P and Q, connected by a string

passing over a fixed pulley, be in equilibrium on a double inclined plane, show that

$$\frac{P}{Q} = \frac{\cos\theta}{\cos\phi} \cdot \frac{\sin\beta}{\sin\alpha}$$

where α and β are the angles of the two planes, and θ and ϕ the angles which the two portions of the string make with the planes?

15. In a screw-press the length of the lever is l. What must be the distance between the threads of the screw in order that the greatest possible mechanical advantage of the combination may be M?

16. The mechanical advantage of a screw press, when treated as smooth, is 200, while that of the screw alone is 20; determine the effort which will just overcome a resistance of 260 lb. weight, the coefficient of friction being $\frac{1}{2}$.

17. The distance between the threads of a screw is such that a weight, W, is just sustained without the action of any effort. What is the least effort which will raise the same weight?

CHAPTER XVIII

SIMPLE HARMONIC MOTION

SUPPOSE a particle (or, in fact, any particular point in an extended body) to move on a straight line in such a way that its distance, x, from a point of reference, O, may be written $x = A \cos pt$, where A and p are constants and t is the time; the motion is said to be a *simple vibration*, or to be *Simple Harmonic Motion*.

It will assist the description of the motion to state at the outset that the motion of any point of an ordinary pendulum is very nearly indeed of this kind. The character of the motion may be obtained by examining the expression which defines it.

i. The value of a cosine is never greater than unity or less than minus unity; hence at no time can x be greater than A on either side of the point O. This semi-range, A, which represents the extent of the excursion of the particle on each side of O, is called the *amplitude*.

ii. The value of a cosine is equal to plus unity whenever the angle is an *even* multiple of π, hence x will have the value A whenever $pt = 2\pi . N$, where N is an integer. Between two such times x goes through every value from A to $-$ A and *back* to A. The time interval from when the particle is at the extremity $+$ A to when next it is so is obtained by putting $N = 1$; this interval is called the *time-period.* Denoting it by T we have

$$T = \frac{2\pi}{p}.$$

The number of passages through this extreme point in unit time is the reciprocal of T; it is called the *frequency.* Denoting it by n we have

$$n = \frac{1}{T} = \frac{p}{2\pi}.$$

Since the constant, p, is proportional to the frequency it is called the *frequency-constant*; it is equal to $2\pi n$ or $2\pi/T$.

It should also be noted that the cosine of an angle in general goes through any one of its values twice during an increase of the angle by 2π; one of these times the cosine is increasing as the angle is passed through, the other time it is *decreasing.* Between two successive passages through any position, if the displacement *increases in both cases as it passes through,* the angle pt increases by 2π, and the corresponding time interval is therefore the time-period.

The equation, $x = A \cos pt$, expresses that x is at a maximum value, A, when t is zero. Exactly the same sequence of values may be passed through without reckoning the time from that corresponding to a maximum value. The more general expression of simple harmonic motion (S. H. M.) is $x = A \cos (pt + e)$, where e is a constant, which expresses that when t is zero $x = A \cos e$. The constant, e, is called the *epoch*, or the *angle of lead*; the negative of it is called

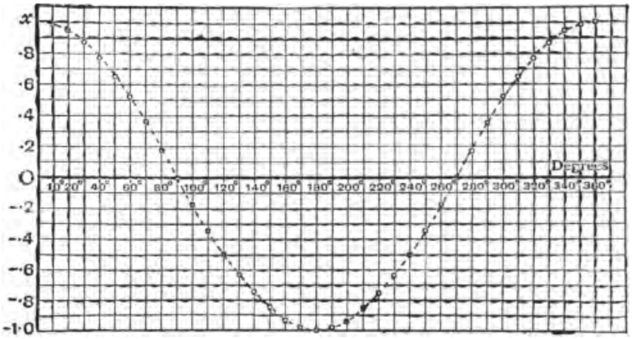


Fig. 117.

the *lag*. The angle $(pt + e)$ is called the *phase*; therefore the angle of lead is the initial phase. In particular e may equal $-\pi/2$; in which case $x = A \cos (pt + e)$ becomes $x = A \sin pt$.

The sequence of changes in x may be depicted diagrammatically by plotting x against t, numerical values of the constants A, p, and e, being assumed. For example, take $e = o$, $p = \pi/18$, and $A = 1$. Unit increase in time then increases pt by $\pi/18$ radians, *i.e.* by 10°; hence, from a book of mathematical tables, pick out the values of the cosine for every 10°, and plot them against $t = 1, 2, 3$, etc. A curve similar to Fig. 117 will be obtained.

Examination of the motion of any point on a pendulum will show that it slows down to rest at each extremity of the path, while the velocity is greatest in the middle. Remembering that the velocity is given everywhere by the *slope* of the displacement curve, these facts could be ascertained from the above diagram by finding its slope at various points. We shall now show how to determine the velocity directly from the equation.

Let the value of x corresponding to time t_1 be written x_1, and to t_2 be written x_2.

Then
$$x_2 = \text{A} \cos (pt_2 + e),$$
$$x_1 = \text{A} \cos (pt_1 + e).$$

Subtract and divide by $t_2 - t_1$

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{\cos (pt_2 + e) - \cos (pt_1 + e)}{t_2 - t_1}.$$

This ratio is the average velocity in the interval $t_2 - t_1$; its value could be found after inserting numerical values, if the values of the cosines were ascertained by tables. The velocity at any instant is the limiting value of this expression, as t_2 is made to approach t_1. We shall make use of the following trigonometrical theorems:—

$$\text{Cos} (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.$$
$$\text{Sin} (\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi.$$

$$\text{Cos}\ \phi = 1 - \frac{\phi^2}{2} \text{ nearly when } \phi \text{ is small.}$$

$$\text{Sin}\ \phi = \phi - \frac{\phi^3}{6} \text{ nearly when } \phi \text{ is small.}$$

Hence, with the same restriction, writing $p(t_2 - t_1) = \phi$.

$$\frac{x_2 - x_1}{t_2 - t_1}$$

$$= \mathrm{A}\frac{\cos(pt_1 + e + \phi) - \cos(pt_1 + e)}{t_2 - t_1}.$$

$$= \frac{p\mathrm{A}}{\phi}\Big[\cos(pt_1 + e)\cos\phi - \sin(pt_1 + e)\sin\phi - \cos(pt_1 + e)\Big]$$

$$= \frac{p\mathrm{A}}{\phi}\Big[\Big(1 - \frac{\phi^2}{2}\Big)\cos(pt_1 + e) - \Big(\phi - \frac{\phi^3}{6}\Big)\sin(pt_1 + e) - \cos(pt_1 + e)\Big]$$

$$= -p\mathrm{A}\sin(pt_1 + e) + \text{terms which contain } \phi \text{ in the numerator,}$$

and which therefore vanish when ϕ is made indefinitely small.

Hence the velocity v is given by

$$v = -p\mathrm{A}\sin(pt + e).$$

Now the sine of an angle is zero when the cosine is at a maximum (positive or negative), and at a maximum (positive or negative) when the cosine is zero; hence the velocity is zero when the displacement is numerically greatest, and *vice versâ.*

We proceed to find the acceleration in a precisely similar way. Let v_2 and v_1 be the velocities at two neighbouring times t_2, and t_1, differing by ϕ/p; then

$$\frac{v_2 - v_1}{t_2 - t_1}$$

$$= -\ p\mathrm{A}\left\{\frac{\sin(pt_2 + e) - \sin(pt_1 + e)}{t_2 - t_1}\right\}$$

$$= -\frac{p^2A}{\phi}\left\{\sin(pt_1 + e)\cos\phi + \cos(pt_1 + e)\sin\phi - \sin(pt_1 + e)\right\}$$

$= -p^2A\cos(pt_1 + e)$ + terms which contain ϕ in the numerator, and which vanish when ϕ is made indefinitely small.

Therefore the acceleration a is given by the equation,

$$a = -p^2A\cos(pt_1 + e),$$

and this may be written $a = -p^2x$, or in words:—

The acceleration when motion is simple harmonic is proportional to the displacement, but of opposite sign. Thus, if we reckon to the right of O as positive, when x is to the right the acceleration is directed to the left—*i.e.* toward the centre, O; when x is to the left the acceleration is directed to the right—*i.e.* again to the centre. Hence, in S.H.M., the acceleration is always directed towards the centre. It should be noted that p^2—from which the time period can be calculated—is the numerical value of the acceleration when the displacement is unity.

If m is the mass of the particle the force acting upon it (producing the motion) is ma—that is, $-p^2mx$; the force then is also proportional to the displacement, but of opposite sign, indicating that it too is always directed toward the centre. Simple harmonic motion is often defined as being the motion due to such a force.

These results can also be ascertained by a simple geometrical method.

Let the path of the particle be made the diameter C′OC, of a circle of radius, A. Draw any other radius $\overline{\mathrm{OM}}$ and drop a perpendicular $\overline{\mathrm{MN}}$ on $\overline{\mathrm{OC}}$. Then $\overline{\mathrm{ON}} = \overline{\mathrm{OM}} \cos \theta$, and since $\overline{\mathrm{OM}}$ is constant the motion of N will be S. H. M. if θ increases proportionally to the time: let $\theta = pt$ accordingly. But if θ is proportional to the time the point M moves uniformly round the circle, for the arc $\overline{\mathrm{MC}}$ is proportional to θ. The point M is called the auxiliary point, because from the properties of its motion we can infer those of N; in fact, the velocity and acceleration of N are the components in the direction OC of the velocity and acceleration of M.

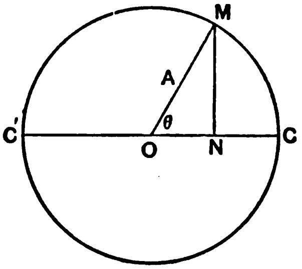


Fig. 118.

Since M moves uniformly round the circle with angular velocity $\theta/t = p$, its acceleration is directed toward the centre (*i.e.* from M towards O) and is equal to $\mathrm{A}\theta^2/t^2$ or $\mathrm{A}p^2$ (p. 97).

Now an acceleration in the direction $\overline{\mathrm{MO}}$ is the resultant of one along $\overline{\mathrm{MN}}$ and one along $\overline{\mathrm{NO}}$, and these three lines are proportional to the three accelerations. Hence the component in the direction $\overline{\mathrm{ON}}$ is

$$-\mathrm{A}p^2\frac{\overline{\mathrm{ON}}}{\overline{\mathrm{OM}}} = -\mathrm{A}p^2 \cos pt = -p^2 x,$$

and this is the acceleration of N as was found before. Similarly the triangle ONM is a triangle of velocities *if it be turned round through a right* *angle*, so that $\overline{OM}$ is parallel to the tangent at M (and $\therefore$ parallel to the velocity $A\frac{\theta}{t} = Ap$ of M); moreover $\overline{NM}$ is then parallel to $\overline{OC}$. The component parallel to OC is

$$-Ap\frac{\overline{MN}}{\overline{OM}} = -Ap\sin pt,$$

and this is the velocity of N.

It has been shown that in S. H. M. the external force is given in terms of the displacement, by the equation $F = -p^2mx$. Since $p = 2\pi/T$ the equation for the time period of the motion is

$$T = 2\pi\sqrt{-\frac{mx}{F}}$$

$= 2\pi$ divided by the square root of the numerical value of the acceleration at unit distance from the centre.

Simple Pendulum.—Let the bob of a simple pendulum be withdrawn so that the string makes a *small* angle with the vertical. The forces on the bob are its weight, W, and the tension, T, in the string. The string constrains the bob to move in a circular path the component force along which is $-W\sin\theta$. If the angle is small the path itself may be taken as coinciding with its horizontal component x,

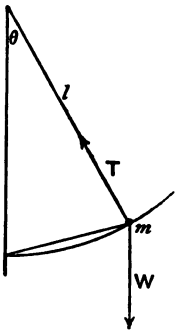


Fig. 119.

and the force as being $-W\theta$ or $-W\frac{x}{l}$. Hence, force and displacement x are, to this approximation, proportional to one another, and the motion is to the same extent S. H. M. The equation to the motion is

$$x = A \cos pt$$

where A is the greatest value of x, and

$$p = \frac{2\pi}{T} = \sqrt{\frac{W}{ml}} = \sqrt{\frac{g}{l}}.$$

The time period is $2\pi\sqrt{l/g}$.

The velocity of the bob is

$$-pA \sin pt = -A\sqrt{\frac{g}{l}}.\sin pt.$$

This value should be compared with that determined on p. 121 by means of the principle of energy.

The angular velocity of the string is

$$-\Theta\sqrt{g/l}.\sin pt$$

where Θ is the greatest value of the angular deflection.

This angular velocity must be very carefully distinguished from the angular velocity p, of the auxiliary point made use of in the alternative method of deducing the properties of the motion.

Extension of Definition.—Any two quantities are said to be connected simple harmonically when one varies as the cosine or sine of a multiple of the other. The characteristics of the variation are the same as those of the motion that we have discussed. The kind most nearly related to this is that in which the quantities are couple and angle respectively. Such motion occurs in the case of a rigid body which is suspended at a fixed point, and is given a *small* initial displacement from its equilibrium position. A body so suspended is called a *compound* or *physical pendulum*.

Compound Pendulum. Let the body be as shown in Fig. 120, and supposed to be free to move in the plane of the paper about the fixed point, S (the point of suspension), distant b from the centre of mass, G. The external forces are the weight, W, and the action at the support. The moment of these forces about S is $-Wb \sin \theta$, where $\overline{SG}$ makes an angle θ with the vertical—the minus sign indicating that it opposes the increase of θ. If the angle is small this equals $-Wb.\theta$ nearly; therefore the couple and the angle are proportional to one another and of opposite sign; the angular motion is therefore S. H. M.

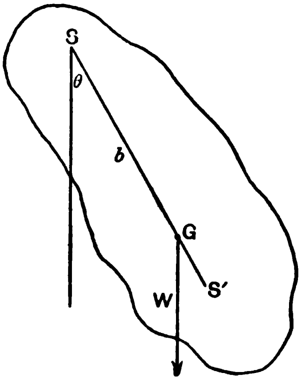


Fig. 120.

The time period can be calculated, if the value of p be determined. For linear displacements we saw that p^2 is the acceleration when the displacement is unity. In the present case we are concerned with angular acceleration and angular displacement, and by analogy p^2 equals the angular acceleration when the angular displacement is unity. Now the moment of the forces about S is numerically equal to $Wb\theta$; but it is also equal to $I_s\mathbf{a}$ where I_s is the moment of inertia about S and $\mathbf{a}$ is the angular acceleration; hence $p_s = Wb/I_s$ and

$$T = 2\pi/p = 2\pi\sqrt{\frac{I_s}{Wb}}$$

If κ_0 is the spin-radius with respect to the centre of mass

$$I_S = m(\kappa_0^2 + b^2) \text{ and } T = 2\pi\sqrt{\frac{m(\kappa_0^2 + b^2)}{mgb}}$$
$$= 2\pi\sqrt{\frac{\kappa_0^2 + b^2}{gb}},$$

or the time period is the same as that of a simple pendulum whose length is $(\kappa_0^2 + b^2)/b$; we will denote this length by L; it is called the *length of the equivalent simple pendulum.* Select a point S′ on $\overline{SG}$ produced, such that $\overline{SS'} = L$; the point S′ is called the *centre of oscillation;* it is such that if the whole mass were concentrated there, and connected with S by a massless string, the time period would be unchanged.

It should be noticed that T is the same, provided b remains unchanged; hence the time period is the same for all positions of S on a circle of radius b and centre G. We will now prove that it has also the same value for all points on a circle of radius $\overline{\mathrm{GS}'}$ and centre G. Let $\overline{\mathrm{GS}'} = b'$. Then if S′ is made the point of suspension the time period is

$$\mathrm{T}' = 2\pi\sqrt{\frac{\kappa_0^2 + b'^2}{b'g}}$$

$$\text{but } b' = \overline{\mathrm{SS}'} - b = \frac{\kappa_0^2 + b^2}{b} - b = \frac{\kappa_0^2}{b}$$

$$\text{hence } (\kappa_0^2 + b'^2)/b' = \frac{\kappa_0^2}{b'} + b' = b + \frac{\kappa_0^2}{b} = \frac{\kappa_0^2 + b^2}{b}$$

$$\therefore\ \mathrm{T}' = 2\pi\sqrt{\frac{\kappa_0^2 + b^2}{bg}} = \mathrm{T},$$

that is the time period is the same whether S′ or S is the point of suspension, and since it is also the same as long as b' is unchanged, the theorem is proved. It must be repeated that the motion is assumed to be confined to one plane; if motion can take place in any other plane the period will be different, in general, owing to the different value of I_s.

It is easy to see from the formula that when b is infinite the time period is infinite, and that it also is so when $b = o$. Between the two it must at some value of b fall off to a minimum. In order to find the value of b which corresponds to this minimum it may first be noted that T is least when $(\kappa_0^2 + b^2)/b$ is least, and therefore when $\dfrac{\kappa_0^2 + b^2}{2b\kappa_0}$ is least. Now construct a right angled triangle (Fig. 121) whose sides are b, κ_0, and $\sqrt{\kappa_0 + b^2}$. If ϕ is the angle opposite b

$$\frac{\kappa_0^2 + b^2}{2b\kappa_0} = \frac{1}{2 \sin \phi \cos \phi} = \frac{1}{\sin 2\phi}$$

But $\sin 2\phi$ is greatest when $\phi = 45°$ and b must then equal κ_0; or

The least value of T corresponds to the case in which b equals the spin-radius about the centre of mass.

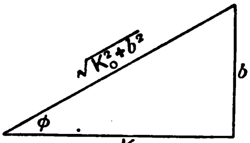


Fig. 121.

The way in which T^2 varies as b is changed is shown in Fig. 122; the ordinates are proportional to T^2 and the abscissæ to b.

The principle of interchangeability of S and S′ (or of "reversibility of the pendulum" as it is commonly called) is made use of in the accurate determinate of g. A bar pendulum is taken, having two knife-edges, by each of which it may be suspended in turn. One of the knife-edges can be

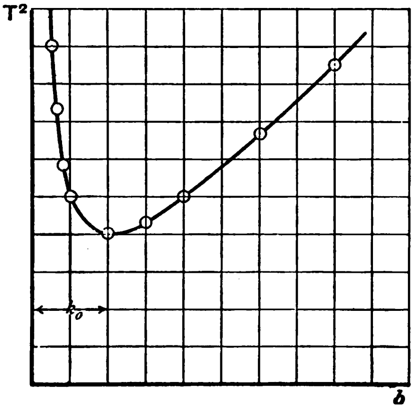


Fig. 122.

slid along the bar and a position is sought, for which the period is the same as for the fixed one. The distance between the knife-edges is then the length of the equivalent simple pendulum, provided that they are on opposite sides of the centre of mass; and g is then calculated from the equation $T = 2\pi\sqrt{\frac{L}{g}}$. The advantage of this method consists in that the distance between the knife-edges can be more accurately measured than the length of a simple pendulum: indeed,

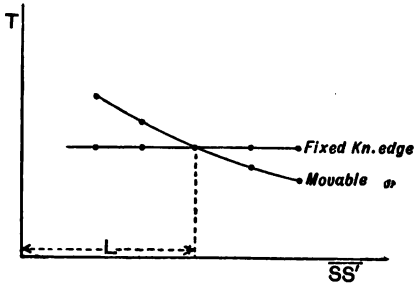


Fig. 123.

the accuracy obtainable is very great. The tedious readjustments, necessary to obtain practically accurate accordance of the time periods, can be avoided in either of two ways. If a series of values of the time period about each knife-edge for various distances between them be obtained and plotted—on the same diagram—against the distance between them the point of intersection of the curves corresponds to the value of SS′ which gives accordance of periods.

The second way is to transform the formula. The time periods for any two points are given by

$$T_1^2 b_1 = 4\frac{\pi^2}{g}(\kappa_0^2 + b_1^2) \qquad T_2^2 b_2 = 4\frac{\pi^2}{g}(\kappa_0^2 + b_2^2)$$

whence

$$T_1^2 b_1 - T_2^2 b_2 = 4\frac{\pi^2}{g}(b_1^2 - b_2^2).$$

The left-hand side can be written

$$\frac{T_1^2 + T_2^2}{2}(b_1 - b_2) + \frac{T_1^2 - T_2^2}{2}(b_1 + b_2),$$

hence $$\frac{T_1^2 + T_2^2}{2} + \frac{T_1^2 - T_2^2}{2}\,\frac{b_1 + b_2}{b_1 - b_2} = \frac{4\pi^2}{g}(b_1 + b_2).$$

If the points of suspension are on opposite sides of the centre of mass, $b_1 + b_2$ equals the distance between the knife-edges. The formula is perfectly accurate, but the denominator $b_1 - b_2$ is not obtainable with the same experimental accuracy as $b_1 + b_2$. If, however, the periods are nearly accordant the second term on the left is small and a fairly rough value of it is sufficient for giving considerable accuracy in the value of g calculated by means of the equation. The value of $b_1 - b_2$ can be obtained by balancing the bar on a glass rod and measuring from each point of suspension to the point of contact between the rod and bar.

Centre of Percussion.—When a compound pendulum receives a blow at any point there is, in general, a reaction called into play at the point of support; for the blow would usually

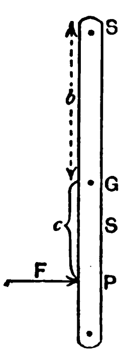
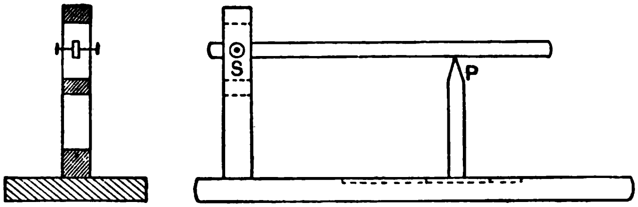


Fig. 124.

set S in motion if it were free to move; the force called into play at the fixed support must be at each moment just sufficient to prevent this motion. For example let a force, F, act for a short time, t, at a point, P, distant c from the centre of mass, G (Fig. 124). The effect of this impulse is twofold. It generates a translational velocity of the whole body equal to Ft/mass; it also produces an angular velocity round the centre of mass equal to Ftc/I, where I is the moment of inertia about G. Owing to the rotation the point S moves at the rate $bFtc/I$, and this will be in the same or in the opposite direction to the translational component according as P is above or below the centre of mass. We can therefore choose a value for c when P is below G, such that S does not move at all; this will be when $\frac{bFtc}{I} = \frac{Ft}{\text{mass}}$, or when $c = \frac{k_0^2}{b}$ where k_0 is the spin-radius about the centre of mass. Since S receives no motion in this case no reaction will be called out at S. The point so determined is called the *centre of percussion*; it is the point at which the body should be struck in ballistic experiments for determining the velocity of a shot. It should be observed that it is the same point as that to which we previously gave the name *centre of oscillation*. The position of the point can be determined experimentally by means of the apparatus shown in Fig. 125. The bar is loosely pivoted at S; the blunt knife-edge, P, can be altered in position. If the bar be lifted



Cross Section Side View

Fig. 125.

at the point above P and then allowed to drop, the pivoted end will break loose from its support unless P is near the centre of percussion. If P is too near S, the end at S will be jerked upwards by the blow; if it is too far away, the jerk will be downwards.[1] The velocity which S would receive, if it were free to move, whether P is at the centre of percussion or otherwise, is

$$\frac{Ft}{m} - \frac{Ftcb}{mk_0^2} = \frac{B}{m}\left(1 - \frac{cb}{k_0^2}\right),$$

where B is written for F*t*, the blow applied at P. The blow B′, which applied at S would exactly prevent this velocity arising, is one which, acting alone, would create an equal opposite velocity at S; *i.e.* it is given by the equation

$$\frac{B'}{m} + \frac{B'b}{mk_0^2} = \frac{B}{m}\left(1 - \frac{cb}{k_0^2}\right); \text{ whence } B' = B\frac{k_0^2 - cb}{k_0^2 + b^2}$$

[1] An old specimen of this apparatus, made by Dunn, of Edinburgh, in 1828, exists among the apparatus in the Physical Laboratory, University College.

This is the measure of the impulsive reaction at the support, S.

Ball Oscillating on Concave Surface. A bicycle ball (radius r) rolls backwards and forwards on the concave surface of a lens of radius $R + r$, placed horizontally. Find the time period of a small oscillation. Here the component of the forces in the direction of the path is $ma = -mg \sin \theta + F$, where F is the force of friction. The friction is supposed to be great enough at each instant to prevent any slip at the surface of contact. Its value at any instant must therefore be such as to preserve a constant relation between the linear and angular accelerations. This relation can be found from the consideration that while the ball rolls a distance on the lens equal to its own circumference, it rotates about its centre of mass through an angle $2\pi - \theta$; where

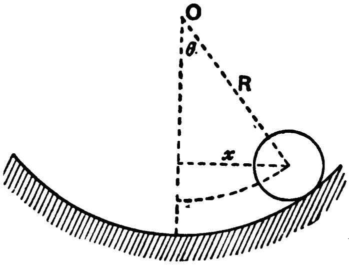


Fig. 126.

$$\theta = \frac{\text{circumference}}{R + r} = \frac{2\pi r}{R + r}.$$

In the same interval the centre of mass has travelled a distance equal to $R\theta$; therefore

$$\frac{\text{Linear displacement of centre of mass}}{\text{Angular displacement round centre of mass}} = \frac{R\theta}{2\pi - \theta} = r$$

But this must also be the ratio between the linear and angular accelerations.

Now the angular acceleration, **a**, is $-\mathrm{F}r/\mathrm{I}$ and

$$a = r\mathbf{a} = \frac{-\mathrm{F}r^2}{mk_0^2}.$$

Hence $$ma = -mg\sin\theta - \frac{mak_0^2}{r^2}$$

and $$a = -\frac{g\sin\theta}{1 + k_0^2/r^2}.$$

If the angle θ is always small we may write x/R instead of $\sin\theta$,

or $$a = -\frac{gx}{\mathrm{R}\left((1 + \frac{k_0^2}{r^2}\right)}.$$

Since a is proportional to x, and is positive when x is negative, the motion is a Simple Vibration (S. H. M.) and the time period is

$$2\pi/(\text{accel. for unit } x)^{\frac{1}{2}} = 2\pi\sqrt{\frac{\mathrm{R}\,(1 + k_0^2/r^2)}{g}}.$$

For a sphere $k_0^2 = 2r^2/5$, hence

$$\mathrm{T} = 2\pi\sqrt{\frac{7\mathrm{R}}{5g}}.$$

Hence the length of the equivalent simple pendulum is $\frac{7}{5}$ × difference of radii of lens and ball.

Composition of Harmonic Motions. When two forces act together, each of them being such as acting alone would produce simple harmonic motion along a given straight line, the motion in general becomes more complicated. A particularly simple case arises, however, when the separate motions have the same period. Let these separate motions be defined by the equations

$$F_1 = ma_1 = -mp^2x_1$$
$$F_2 = ma_2 = -mp^2x_2$$

In these equations, x_1 and x_2 are displacements which the respective forces F_1 and F_2 would have produced if acting alone; a_1 and a_2 are the corresponding accelerations. The constant, p, is the same for both, for it depends only on the time period.

Now if the two forces are acting together, their resultant is $F_1 + F_2$, the resultant acceleration is $a_1 + a_2$, and the resultant displacement is $x_1 + x_2$.

But the above equations give on addition

$$F_1 + F_2 = m(a_1 + a_2) = -mp^2(x_1 + x_2).$$

Hence, denoting the resultants by letters without suffixes, we have

$$F = ma = -mp^2x;$$

or, in other words, the motion is still simple harmonic, and of the same period as that of each of its components. This may be also seen by adding together the two equations connecting the displacements with the time. Let them be

$$x_1 = A_1 \cos (pt + e_1)$$
$$x_2 = A_2 \cos (pt + e_2).$$

These equations represent two S.H.M.s of the same period, but of different amplitude and phase. The resultant motion is obtained by adding them together:

$$x = x_1 + x_2 = A_1 \cos (pt + e_1) + A_2 \cos (pt + e_2)$$
$$= (A_1 \cos e_1 + A_2 \cos e_2) \cos pt - (A_1 \sin e_1 +$$
$$A_2 \sin e_2) \sin pt = C \cos pt - S \sin pt.$$

Now if an angle η be chosen such that

$$\tan \eta = \frac{A_1 \sin e_1 + A_2 \sin e_2}{A_1 \cos e_1 + A_2 \cos e_2} = \frac{S}{C}$$

the above equation may be written

$$x = \sqrt{S^2 + C^2} \cos (pt + \eta);$$

and this is the equation to another S.H.M. of the same period, but of different amplitude and initial phase.

When the periods of the components are not the same, the one that has the shorter period is always gaining on the other as regards phase; so that if at any moment the phase of each is the same it will not continue so. The result is that at some moments they help one another in increasing the displacement, at other moments they oppose one another, the displacement due to one being positive while that of the other is negative. The resultant displacement then ceases to have the same simple character.

Energy of a S.H.M. The kinetic energy of a particle of mass, *m*, executing a simple vibration can be obtained from a knowledge of the velocity.

Let the motion be $x = \mathrm{A} \cos (pt + e)$; then we have seen that the velocity is $v = - \mathrm{A}p \sin (pt + e)$. Hence the kinetic energy is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mathrm{A}^2p^2 \sin^2 (pt + e).$$

Thus it undergoes fluctuations; but it should be carefully noticed that it does not vary in a simple harmonic way; it is at each instant proportional to the square of a sine and not to the sine itself. This energy is greatest when $pt + e =$ an odd multiple of $\pi/2$, for the sine then equals unity, *i.e.*

$$\text{Max. kinetic energy} = \tfrac{1}{2}m\mathrm{A}^2p^2 = \frac{2\pi^2 m\mathrm{A}^2}{\mathrm{T}^2}.$$

On the other hand, whenever $pt + e$ equals an even multiple of $\pi/2$ the kinetic energy is zero. Since the force which produces a simple vibration depends only upon the position of the body we should expect, as shown in p. 232, that potential energy will result whenever work is done against it; and that the total amount of energy of both kinds will be invariable. Considering the total potential energy as zero when the body is in its central position, this statement implies that at either of the extreme positions, where the kinetic energy is zero, the potential energy must have a maximum value of $\tfrac{1}{2}m\mathrm{A}^2p^2$; and further that, in any position, this expression gives the total quantity of energy of both kinds. In the case of a pendulum, the potential energy is due to gravity.

EXAMPLES TO CHAPTER XVIII.

1. The period of the simple harmonic motion of a point is 20 sec., and its maximum velocity is 10 ft. per sec. Find its velocity at a distance of 20 ft. from the mean position.

2. A particle executing simple harmonic motion moves with a velocity of 10 cm. per sec. through its equilibrium position, and has an amplitude of 4 cm. Calculate the time period.

3. If the amplitude of a simple harmonic motion is 20 cm., and the maximum velocity is 4 cm. per sec., calculate the velocity when the displacement is 15, 10, and 5 cm. respectively.

4. The mass of the moving body in Example 3 being 10 grams, calculate the kinetic energy at the positions 0, 5, 10, 15, 20 cm. respectively.

5. Two simple harmonic motions in the same straight line and of the same period have amplitudes of 10 and 20 cm. respectively. If they both affect the same particle, find the actual amplitude of the latter's motion in the following cases: (i) If they agree in phase; (ii) If they differ in phase by 45°; (iii) If they differ in phase by 90°.

6. In Example 5 find the relation between the maximum kinetic energy of each separate motion and of the resultant motion.

7. A spiral spring hangs vertically with a weight attached. Assuming that the force required to stretch the spring is proportional to

the extension, find the time period of the oscillation which takes place if the weight is pulled down and then let go, being given the following data:

Mass of weight = 100 grams.

An addition of 10 grams causes the spring to stretch 2 cm. (Neglect mass of spring.)

CHAPTER XIX

MECHANICS OF FLUIDS

THE bodies considered so far have been those whose parts do not change their relative positions; or, if they do so, change them so slowly that they can be treated as though they were rigid. In this chapter we deal with *fluids*: these may be defined as substances which can exert no permanent resistance to change of shape; so that if, for example, some is placed in a vessel, it ultimately, under the influence of its own weight, acquires the shape of the vessel, or at least, of that portion which it occupies. Water, alcohol, treacle, are examples of fluids which exert considerable resistance to a decrease of *volume*—they are distinguished under the name *liquids*; air, hydrogen, and coal gas are examples of fluids which, under usual circumstances, exert comparatively little resistance to a decrease of volume and are called *gases*.

If a gas is admitted into a closed empty vessel it at once expands and fills it; if a liquid is admitted into a vessel in insufficient quantity to fill it, only the lower portion of the vessel is occupied by liquid, the volume of the latter remaining practically unchanged. Liquids, like solids, are indeed so slightly compressible that except in special cases, it is unnecessary to allow for the variation in volume that may take place. On the other hand, the volume of a given mass of gas changes so conspicuously, when the forces change to which it is subjected, that it is impossible to ignore the change; indeed, it is often the most important variation produced.

A change in the shape of a substance can, in general, be brought about only by the sliding of one layer of it over another. In a fluid it is this change which meets with no *permanent* resistance, so that, if the external forces tend to change the shape, they ultimately will. In every real fluid there is a *temporary* resistance to such change. This opposing force is a frictional one; but it does not follow the laws of the friction of solids.

Its characteristic is that it depends upon the gradual relative velocity of the moving layers, and vanishes when all relative motion ceases. So that, although considerable resistance to sliding is called out when the rate of change of velocity with distance is great, yet, because it acts so as to oppose this relative motion, it must continuously grow less as the motion itself decreases, and when the latter is vanishingly small becomes negligible.

The ideal fluid which would experience no such resistance in whatever way it moved is called a *perfect fluid*. No such fluid exists; but from what we have just said, it will be seen that when a fluid has come to rest, the relations between the forces acting upon it must be precisely the same as if it were perfect.

Pressure at a Point in a Fluid.—If a body be immersed in a fluid, every portion of its surface experiences, in general, a force due to the surrounding fluid. If both body and fluid are at rest, this force, unlike that between two solids in contact with one another, is everywhere at right angles to the boundary between them. This follows at once from the fact that, if there were any component *parallel* to the surface of separation, it would at once cause the water in the immediate neighbourhood to slide over the surface; for its motion would be unresisted by any force due to the fluid surrounding it. This is true, whether the surrounding fluid is a perfect fluid or not, for friction in all fluids becomes vanishingly small as the relative velocity between moving layers of it also becomes vanishingly small. The force in all usual cases is a pressure; that is, the forces on the boundary of any portion of the fluid act so as to compress it.

The *average pressure* over any plane surface in a fluid is obtained by dividing the total force on the surface by the area. The limiting value of this ratio, as the area is made indefinitely small, is called the pressure at a point contained in the area; we shall denote it by a small letter p.

A principle, which is made extensive use of in hydrostatics, is the principle of separate equilibrium. If a body of fluid is in equilibrium, each part of it must be so also; therefore, we may select any particular part of it, and apply the conditions of equilibrium to it. The forces on such selected portion will consist of its weight, together with those forces which act on its surface and are due to the contact between it and the surrounding fluid. Now the portion selected is fluid, and the student may feel some uneasiness in applying the conditions which have only been proved in connection with a rigid body to the equilibrium of a fluid. But it should be pointed out that, even if additional conditions are necessary, those applying to a rigid body must at any rate be satisfied. For, if each particle is in equilibrium, the resultant force on it must be zero, and therefore the vector sum of such resultants must be zero. Again, if there is to be no rate of change of moment of momentum (p. 215), the resultant moment of the forces must be zero. Thus both the conditions applying to a rigid body must be satisfied. In order to relieve the difficulty which may be felt, the selected portion is sometimes imagined to become solidified *without changing its density.* Such a solidified portion would still be in equilibrium, and since the forces upon it are not changed by the solidification, whatever relation holds between them must hold prior to the solidification. Since when it has become solid it is a rigid body, the conditions applicable to such a body can be applied without hesitation. But the student is urged to try and realise that such an imagined solidification is somewhat artificial and unnecessary.

Since pressure is a force per unit area it can be expressed in terms of dynes per sq. cm., or pounds weight per sq. in., or per sq. ft.; or indeed, in terms of *any* unit of force per *any* unit of area. The mode of transferring from one unit to another will be clear from the following examples:

$$x\,\frac{\text{lb. wt.}}{\text{sq. in.}} = x\,\frac{\text{lb. wt.}}{\text{sq. ft.}/144} = 144x\,\frac{\text{lb. wt.}}{\text{sq. ft.}}$$

$$= x\,\frac{453{\cdot}6\times 981\text{ dynes}}{(2{\cdot}54\text{ cm.})^2} = x\,\frac{453{\cdot}6\times 981}{2{\cdot}54\times 2{\cdot}54}\,\frac{\text{dynes}}{\text{cm.}^2}$$

Independence of Aspect.—*The value of the pressure is independent of the aspect of the surface* to which it relates. To prove this theorem consider the equilibrium of a right-ended horizontal prism of the fluid. Such a prism has three lateral faces and two ends. The equilibrium is maintained by the forces on these surfaces due to the surrounding fluid, acting together with the weight of the prism itself. The latter acts through the

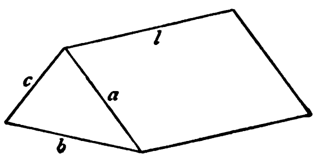


Fig. 127

centre of gravity, which lies in the mid section of the prism. The forces on any lateral face can be replaced by a single force, normal to the surface and equal to their resultant; and this, if the surface is so small that the pressure can be considered to have the same value at every point of it, is proportional to the area and acts at the mid point of the surface, and therefore in the mid section of the prism. Let the lengths of the sides of this mid section be *a*, *b*, *c*, and the length of the prism be *l*. Let the pressures for points on the three lateral surfaces be p_1, p_2, and p_3; the forces on the same faces are then p_1al, p_2bl, p_3cl, respectively. Since all four of the above forces act at points in the mid section, we will represent this section in a separate figure (Fig. 128). The forces on the *ends* of the prism act parallel to the length, and since we only intend considering the equilibrium as regards *lateral* motion, their values need not be calculated.

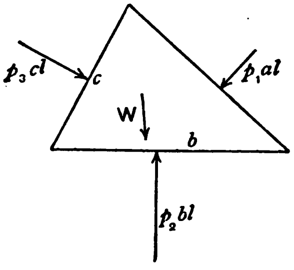


Fig. 128.

Now, of the four forces shown in Fig. 128, three are proportional to the areas of the surfaces of the prism, and the fourth (the weight W) is proportional to the volume. We desire to find a relation between these forces *in the limiting case when the prism is made indefinitely small.* Let, therefore, the prism be taken successively smaller and smaller, the length remaining constant, and the cross-section remaining similar throughout. The areas of the surfaces will decrease proportionally to the linear reduction, but the *volume* will decrease proportionally to the square of it. It follows therefore that, however large W may be at the commencement, compared with any one of the forces on the surfaces, it will at last, since its diminution goes on at a faster rate, become smaller than any of them; and ultimately, when the mid section has dwindled to a point, it will become vanishingly small in comparison, and may therefore be neglected. For example, suppose that initially the weight is equal to one of the surface forces. If each side is reduced to one-tenth, the new weight will be one-tenth of the new surface force; if each side is reduced to one-millionth the weight is one-millionth of the new surface force, and so on till it vanishes in comparison.

There remain the three forces p_1al, p_2bl, p_3cl, and these must therefore be in equilibrium with one another. Now they act at the mid points of the sides of the section shown, and are normal to these sides. If the section be turned through a right angle in its own plane, its sides will become parallel to the forces; they are therefore also proportional to them; that is to say,

$$\frac{p_1al}{a} = \frac{p_2bl}{b} = \frac{p_3cl}{c},$$

or

$$p_1 = p = p_3.$$

Thus when the section has become indefinitely small, so that the sides represent three surfaces of different aspects (*i.e.* facing in different directions), but all passing sensibly through the same point, the value of the pressure (or force per unit area) is the same for each surface.

Since the prism selected may have a cross-section of any triangular shape, the same result must be true for surfaces facing in any direction in the plane of the above section.

By keeping the aspect of one surface fixed, and placing the *length* of the prism in all directions successively, the same result can be shown to be true for any aspect whatever.

It is impossible to overestimate the importance of this result; the simplicity of the mechanics of a fluid at rest is in a large measure a consequence of it. It should be observed that the result is due in part to the forces being necessarily normal to the surfaces in which they act, and consequently it must not be expected to be true except for fluids at rest. No such simplicity exists in the case of a fluid in motion or in that of a solid body.

Variation with Position.—The value of the pressure varies in general from point to point of a fluid at rest. The connection between its values at two points is obtained as follows:—

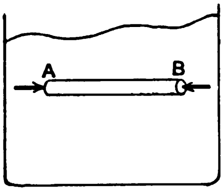


Fig. 129.

i. Two points, A and B, at the same horizontal level.

Consider the equilibrium of a narrow right cylinder of the fluid stretching between the two points. The forces on the curved surface act normally thereto, and have therefore no component along the length of the cylinder; the weight of the cylinder is vertical, and therefore has no component along the length. The forces on the end faces are directed wholly along the length. Since there is no acceleration in this direction these end forces must be equal and opposite. Hence, if p_A is the pressure at A, and p_B the pressure at B, and a the cross-sectional area, we have

$$p_A a - p_B a = 0$$

or

$$p_A = p_B ;$$

that is, the pressure is the same at all points on the same horizontal level of a fluid at rest.

ii. Two points, A and C, on the same vertical, A being the lower.

Consider the equilibrium of a narrow right cylinder of the fluid of cross-section a stretching between the two points. The only forces having vertical components are the forces on the end faces and the weight W of the cylinder, and these are wholly vertical. Hence p_A and p_C being the pressures at A and C respectively,

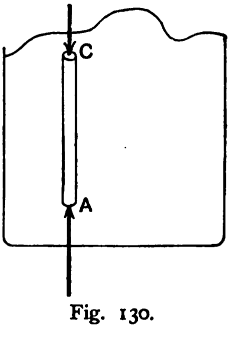


Fig. 130.

$$p_A a - p_C a - W = 0$$
$$p_A = p_C + \frac{W}{a}.$$

Expressed in words, this states that the pressure increases with the depth in a fluid by an amount equal to the weight of a cylinder of the fluid *of unit cross-sectional area* stretching between the two points.

It follows that the difference of pressure at *any* two points in a fluid at rest is equal to the weight of a cylinder of the fluid of unit cross-sectional area, and of a length equal to the *vertical component* of the distance between the points.

All the above theorems are true, both for liquids and gases. The essential difference between the two is that, for ordinary liquids we may consider the density constant, unless enormous forces are applied, whereas the density of gases changes materially under moderate forces.

In the case of a liquid of density ρ

$$\frac{W}{a} = \frac{W \times \text{length}}{\text{vol}} = g\rho \times \text{length} = g\rho h \text{ (say)},$$

where h denotes the difference in vertical height.

Hence $$p_A - p_C = g\rho h.$$

If the C.G.S. system be employed throughout the pressure-difference will be expressed in dynes per square centimetre. When gravitation units are employed, it is better to write the equation in terms of the weight of unit volume instead of the density. The weight per unit volume ($g\rho$) can be expressed in terms of the weight per unit volume of a standard liquid, the ratio of the two being the *specific gravity* with respect to the particular standard. We shall take water as the standard, one cubic foot of which weighs 62·32 lb. wt. $= w$ (say). If one cubic foot of a certain liquid weighs s times this, s is its specific gravity. It is obvious then that $g\rho = sw$, and we may therefore write $p_A = p_C + swh$. Any unit of force may be used in terms of which to express w; if lb. wt./ft.^{3} be employed, and h be expressed in feet, the difference of pressure is expressed in lb. wt. per *square* foot; for

$$sw\frac{\text{lb. wt.}}{\text{ft.}^3} \times h\,(\text{ft.}) = swh\frac{\text{lb. wt.}}{\text{ft.}^2}.$$

If one fluid rests on the top of another of different density without mixing, the surface of separation is horizontal. For consider the pressures at two points, C′, A′, on this surface. They are the same as the pressures at two points, C and A, where C is vertically over A′ and on the same level as C′, and A is vertically under C′ and on the same level as A′.

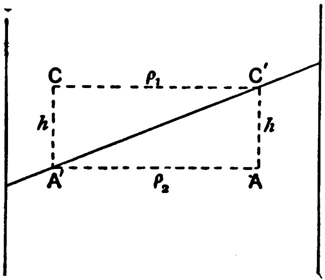


Fig. 131.

Let the distance A′C = AC′ = h, and the densities of the upper and lower liquid be ρ_1 and ρ_2.

$$\begin{aligned} \text{Then } p_A &= p_C' + g\rho_2 h \\ p_A' &= p_C + g\rho_1 h\,; \\ \text{But } p_A &= p_A' \text{ and } p_C = p_C'\,; \\ \therefore\ g\rho_2 h &= g\rho_1 h. \end{aligned}$$

This equation shows that either the densities are the same, or the height, h, is zero; in the latter case, then, A′ and C′ must be on the same horizontal level.

A special case of this is when a liquid does not completely fill a vessel, the remainder of the vessel being empty or filled with a gas. The surface of separation—known as the *free surface*—is then horizontal.

Barometric Pressure.—When the vessel is open to the air the pressure at the free surface is the same as that in the atmosphere at the same level. This is what is indicated by a *barometer*. A mercury barometer consists of a tube filled with mercury and then inverted, without admitting air, and its open end placed in a vessel of mercury. If the tube is long enough—80 cm. say—the mercury sinks and leaves a space above it in the tube, which is vacuous (or at most only contains mercury vapour, which we neglect). This space is known as the Torricellian Vacuum. Since it is

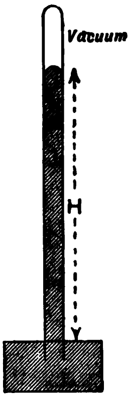


Fig. 132.

vacuous it exerts no pressure on the mercury. The atmosphere presses on the mercury in the vessel; if the value of this pressure is p_0 we have $p_0 = \text{zero} + g\rho H$, where H is the vertical height between the two mercury surfaces; H is known as the Barometric Height, and the formula shows that it is proportional to the pressure of the atmosphere and may be taken as a measure of it, provided that g and p are constants. For accurate work, however, the variations of g with position on the Earth, and of ρ with temperature, must be allowed for before a number is obtained which is strictly proportional to p_0. If g_0 is the value of g in latitude 45° and at sea-level, and ρ_0 is the value of ρ at 0°C, $p_0 = g_0\rho_0H_0$, where H_0 is the height that would have corresponded to the same pressure if g and ρ had been at their standard values. H_0 is called the *reduced height*; its value is given by the equation

$$g_0\rho_0H_0 = g\rho H,$$

or

$$H_0 = \frac{g\rho}{g_0\rho_0}H.$$

In this formula, H is the *true length* at the temperature of observation; if the height is read with a scale correct at 0^0 the expansion of the scale must be allowed for. The acceleration g varies with latitude ($g_\phi = g_{45}(1 - B\cos 2\phi)$ where ϕ is the latitude, $g_{46} = 980{\cdot}62$, $B = {\cdot}002648$), and also with height above the sea level. It diminishes by about ·01 for every 33 metres of ascent.

But "The reduction for elevation is largely a matter of guesswork, and in records of observations the actual values are much more important than the so-called values reduced to sea level" (Everett, C.G.S. Units).

The reduced height is proportional to the pressure; and it is customary to quote it as a pressure of so many centimetres or inches of mercury. The value fluctuates in the neighbourhood of 76 cm., which in most scientific work is taken as the *standard pressure* and is called an atmosphere. The pressure at Greenwich corresponding to this height of mercury is $g\rho H = 981{\cdot}17 \times 13{\cdot}596 \times 76 = 1{\cdot}0138 \times 10^6$ dynes per sq. cm.; that is a little over one megadyne per sq. cm. Convenience in scientific calculation would be promoted if the megadyne per sq. cm. were adopted as the standard.

When the pressure at the free surface of a liquid is known, that at any depth can be calculated. If, for example, it is 1 atmosphere the pressure at a depth of 76 cm. of mercury is 2 atmospheres, at 152 cm. it is 3 atmospheres, and so on. In other words it is the same as though the superincumbent atmosphere were replaced by a column of mercury 76 cm. high, the pressure at the new free surface being zero. This new surface is called the *effective surface.*

Since the pressure due to 76 cm. of mercury is equal to that due to $76 \times 13{\cdot}6$ cm. or $1033{\cdot}6$ cm. of water, the effective surface when p_0 is 1 atmosphere is 1033·6 cm. above the real surface if the vessel contain water. This height is roughly 34 ft.; it would be the Barometric Height for a barometer in which water were used instead of mercury.

The distance h_0 between the free and effective surface is always given by $p_0 = g\rho h_0$; in the above examples p_0 was supposed to be due to a standard atmosphere.

Transmission of Pressure.—If the pressure at the surface of a liquid be increased by any amount the pressure at every point of the liquid will be increased by the same amount. This is known as Pascal's principle.

Its truth is at once evident from the equation,

$$p - p_0 = g\rho h.$$

It must be distinctly understood that it is the pressure and not the force on *any* area which is increased by an *equal* amount.

If two cylinders are connected by a vessel full of liquid, and pistons move up and down them in contact with the liquid, any increase of pressure at one piston is accompanied by an equal increase of pressure at the other. Since the force on each piston is proportional to its area, that piston will experience the greater increase of force which has the larger area. This is the fact made use of in the hydraulic press. By means of a small force applied on a narrow plunger, a large force is brought into play on a plunger of greater cross-section. All attempts to apply this principle were for a long time attended with failure because, owing to the great force, the liquid squirted out round the larger piston instead of raising the piston. This difficulty was first overcome by Bramah in 1796—*i.e.* 145 years after Pascal's enunciation of his principle. In a circular recess cut all round the inside of the cylinder he placed a leather collar in the form of an inverted trough. As the water is squeezed into this by an increase of pressure it is forced into close contact, both with the cylinder and with the surface of the piston. The greater the pressure the closer is the grip of the leather, and the escape of water is thereby prevented.

Force on a Plane Surface.—If a surface in a liquid at rest is plane and horizontal it is easy to calculate the total force upon it; for, since the pressure is in this case the same for all points of it, the total force is pA, where p is the pressure and A the area of the surface.

If the surface is plane but not horizontal the pressure is not the same at all points of it, and the problem must be solved as follows:

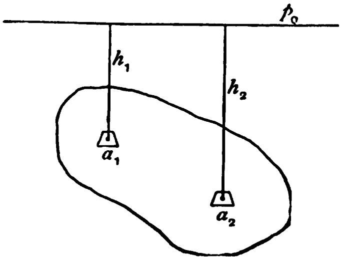


Fig. 133.

The force on any area a_1 of the surface so small that the pressure may be considered uniform over it is pa_1; if this *element* of the surface is at depth h_1 below the free surface at which the pressure is p_0 the force upon it is $(p_0 + g\rho h_1)a_1$. The force on any small element can be found in the same way. Since these forces are all parallel—being normal to the plane surface—their resultant is parallel to them and equal to their sum.

Hence its magnitude is

$$\Sigma[(p_0 + g\rho h)a] = \Sigma(p_0 a) + \Sigma(g\rho ha).$$

Now p_0 is a constant, hence the first term $= p_0 \times$ total area. We proceed to evaluate the second term.

Definition: The product of a small area into its distance from any plane is called its moment with respect to that plane; ha is the moment of the area a, with respect to the free surface of the liquid. If the whole area be supposed gathered together at a point of the surface, such that its moment in the new position is equal to the sum of the moments of its parts in their actual positions, the point at which they are concentrated is called the *centroid* of the area. If H is the distance of this point from the plane of reference,

$$\text{Total area} \times \text{H} = \Sigma(ha).$$

It should be noted that the point so determined is the same as the centre of gravity of a thin uniform lamina coinciding sensibly with the surface. For this reason the *centroid* is frequently called the *centre of gravity of the area*; but the use of this curious phrase should be carefully avoided. Again, the use of the term centroid (*i.e.* centre of form), instead of centre of gravity, when the latter is meant, is equally not to be recommended.

The resultant force on the plane area may now be written

$$(p_0 + g\rho H) \times \text{total area},$$

H being the depth of the centroid of the area below the free surface. The expression in brackets is therefore the *average* pressure at different points of the area: *it is the same as the actual pressure at the centroid.*

If h_0 is the height of the *effective surface* above the free surface, we may write

$$\begin{aligned} \text{Resultant force} &= g\rho(H + h_0) \times \text{total area} \\ &= g\rho H' \times \text{total area}. \end{aligned}$$

From these expressions it is clear that, however large the plane surface may be, the total force on it is not altered in **magnitude** *if the surface be moved in any way which keeps the centroid at the same depth below the free surface.* The direction of the force, of course, turns so as to be normal in each case to the area.

We have spoken of a surface or area in the liquid; this may be either part of the surface of a solid immersed in, or in contact with, the liquid; or it may be an imaginary surface drawn in the liquid. In the latter case the force is that on the liquid at one side of the surface, due to the liquid immediately on the other side of it.

Position of the Resultant. The student must be warned that the resultant force does NOT pass through the centroid *unless* the area be horizontal. We proceed to find the point at which the line of action cuts the area. This point is called the *centre of pressure*; it is, in fact, the centre of the group of parallel forces which act at the various points of the area. Since the points of application of these forces all lie in the plane of the area, their centre does so also.

Now, if a group of parallel forces acting at definite points be turned about these points through any angle, the same for all, the resultant still passes through their centre.

Let the forces then be turned till they all act vertically; each is then proportional to the weight of a cylinder of the liquid standing on the corresponding element of area and extending up to the effective surface (Fig. 134). The resultant force is then represented by the weight of the whole body of liquid vertically above the plane, and imagined to extend to the

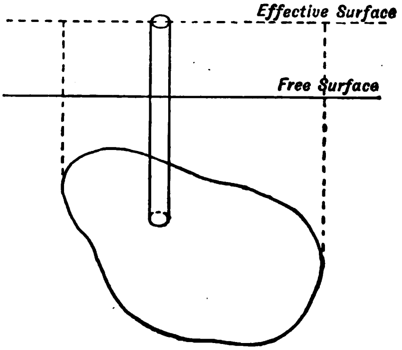


Fig. 134.

effective surface; and the line of action of the weight of this body of liquid (which passes through its centre of gravity) must cut the given plane in the centre of pressure. Hence the centre of pressure is vertically underneath the centre of gravity of the above body of liquid.

Special Cases: (i) Rectangle with one edge in the free surface and $p_0 = 0$ (Fig. 135).

The magnitude of the resultant is given by the previous propositions. Let the sides be as shown, and let θ be the inclination to the vertical. The depth of the centroid is $\frac{1}{2}b \cos \theta$, and the area is ab; hence the total force on the area is $\frac{1}{2}g\rho ab^2 \cos \theta$. The superincumbent volume of liquid forms a wedge; its centre of gravity is on the median lines of the triangular mid-section of this wedge, and is one-third the length of each median from the corresponding side. The vertical through this point cuts the rectangle in a point on the mid-line of the rectangle, which is perpendicular to the side a, and two-thirds its length measured from this side; that is, its depth below the free surface is two-thirds the depth of the lowest side.

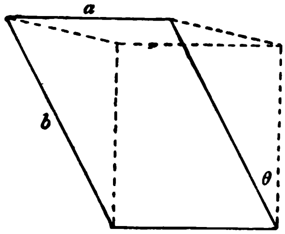


Fig. 135.

Alternative Method: Produce the plane until it cuts the *effective surface*; let it do so in the line XX; this line is called the *line of flotation*.

Let C be the centre of pressure. Now, if we take moments about the line XX, we must have

Moment of resultant
= sum of moments of components.

Divide the area into elements such as Q (Fig. 136); drop perpendiculars from each element to the line XX and also to the effective surface. D is the point in the effective surface which is vertically above Q.

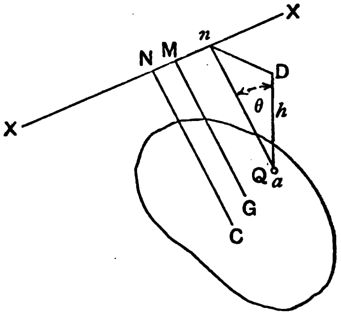


Fig. 136.

If H′ is the depth of the centroid, the resultant is, in magnitude, equal to $g\rho AH'$. Hence

$$g\rho AH' \times \overline{CN} = g\rho\Sigma(ah \times \overline{Qn}).$$

Now the distance of any point on the plane from the effective surface is its distance from the line of flotation multiplied into cos θ where θ is the inclination of the plane to the vertical; that is $H' = \overline{GM} \cos\theta$ and $h = Qn \,.\, \cos\theta$. Hence

$$A \times \overline{GM} \times \overline{CN} = \Sigma(a.\overline{Qn}^2).$$

Definition :—The product of a small area into the square of its perpendicular distance from a line is called the second moment of the area with respect to the line.

Hence, remembering that A × $\overline{GM}$ is the sum of first moments of the area with respect to the line of flotation,

$$\overline{\mathrm{CN}} = \frac{\Sigma(\text{second moments of area})}{\Sigma(\text{first moments of area})}$$

all the moments being taken with respect to the line of flotation.

If instead of an area we had a uniform lamina sensibly coinciding with it, the mass of any element of it would be proportional to its area, and Σ(second moment of area) would be proportional to its ***moment of inertia.*** Hence, the expression is sometimes called the moment of inertia of the area; but this term should be avoided.

If k is the spin-radius with respect to the given line for a lamina coinciding with the plane, and A is the total area.

$$\Sigma(\text{second moment of area}) = \mathrm{A}k^2.$$

$$\text{Hence we may write } \overline{\mathrm{CN}} = \frac{k^2}{\overline{\mathrm{GM}}}$$

The vertical depth of the centre of pressure below the effective surface is $\overline{\mathrm{CN}} \times \cos \theta$.

To find its position on the horizontal line of the plane, which is at this depth, moments would require to be taken about another axis. In this elementary treatise we will only take cases where its position on the line can be judged from symmetry.

Thus, in the case of the rectangle dealt with last the side a is the line of flotation, and the spin-radius about it is $b/\sqrt{3}$; $\overline{\text{GM}}$ is $b/2$, hence $\text{CN} = \frac{2}{3}b$; symmetry of the distribution of the forces on the two sides of the median line (in the vertical plane) of the rectangle shows that C is on this median.

Some other cases can be solved by special methods:

(ii) A triangular plane with one edge in the free surface and $p_0 = o$.

Divide it into narrow horizontal strips of equal width, and select any pair the members of which are equidistant from base and apex respectively. Let c be the width of strip; its length is proportional to its distance from the apex, and the pressure at points on it is proportional to its distance from the base. Hence, force on strip $= g\rho.c \times$ (distance from apex) (distance from base), and this has the same value for both strips; the resultant of these two equal forces passes, therefore, through a point halfway down the median line.

Since the whole triangle can be divided into pairs of such strips, for which the above result is true, the resultant of all the forces on the triangle is at the same point; that is, halfway down the median line.

Pressure on Curved Surfaces

In order to find the resultant force on a curved surface, attention must be paid to the direction as well as to the magnitude of the force on any element. Without considering the elementary forces in detail the horizontal component can be found as follows :—

Consider the equilibrium of the volume of liquid enclosed by the area AB in question, portions such as AC and BD which are horizontal, and the vertical plane whose trace is CD (Fig. 137.)

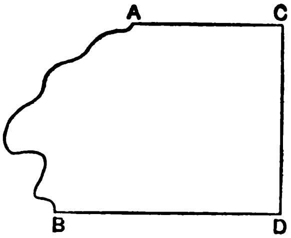


Fig. 137.

Since there is no horizontal acceleration, the sum of the horizontal components of force is zero; hence the horizontal component acting on the curved surface AB is equal to the horizontal component on CD. Now the latter is equal to its area $\times\, g\rho \,\times$ depth of its centroid below the effective surface; this therefore is the horizontal component on AB.

When the surface can be completely closed by a plane lid it is easy to find the total force on it. For the force on the plane lid can be calculated, and this force added by the triangle law to the force on the curved surface must equal the weight of liquid enclosed. We work out a simple example in illustration.

i. Find the force on the curved surface of a hemispherical vessel full of liquid and having its plane side vertical (Fig. 138).

Since each elementary force on the curved surface is normal thereto, each passes through the centre, and so also must their resultant; call this resultant R, and let its direction with the horizontal be denoted by θ. Let the force on the plane surface be X; this force is wholly horizontal. Let W be the weight.

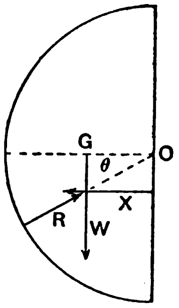


Fig. 138.

Equate vertical components :—

$R \sin\theta = W = \frac{2}{3}\pi r^3 . g\rho$ where r is the radius.

Equate horizontal components :—

$$R \cos\theta = X = g\rho .\ r. \times \pi r^2;$$

since r is the depth of the centroid of the plane surface.

Hence, $\tan\theta = \frac{2}{3}$

$$\text{Or, } \sin\theta = \frac{2}{\sqrt{2^2+3^2}} = \frac{2}{\sqrt{13}},$$

$$\text{And, } R = W/\sin\theta = \frac{\sqrt{13}}{2} W.$$

Indirectly we can find the centre of pressure of a plane circular area. For W acts through a point G, where horizontal distance from O is $\frac{3}{8}r$ (*i.e.*, at the centre of gr. of the enclosed liquid). Hence W and R intersect at a point $\frac{3}{8}r \tan\theta$, or $r/4$ below G. Since X must pass through the same point and is horizontal, it cuts the plane surface at the same depth below O.

Closed Surface. When the surface is completely closed, as it is when a solid is entirely immersed in a fluid, the force on it due to the surrounding fluid must be equal to the weight of the volume of fluid displaced by the solid. For consider the solid removed and replaced by the same volume of the fluid. The latter will be in equilibrium under the action of its own weight and the forces upon it due to the surrounding fluid. These forces must therefore be equal and opposite, and have the same line of action as the weight. But these forces are exactly the same as those acting on the surface of the solid; for both surfaces are the same in size and shape, and each point is in the two cases at the same depth below the effective surface.

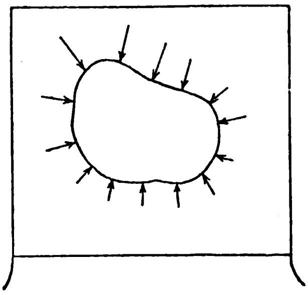


Fig. 139.

The weight of fluid displaced is called the buoyancy: we denote it by B; hence the upward force on the surface of the solid is equal to B. This is known as Archimedes' principle. The line of action of the buoyancy passes through the centre of gravity of the fluid displaced. If W is the weight of the body, this will rise or sink if no other forces are acting, according as W is less or greater than B.

If the body is hung at rest by a thread from a balance arm, the force in the thread is $W-B$; this is called the apparent weight of the body in the fluid.

By weighing a body both in and out of a liquid, the ratio of the specific gravities of the two may be found.

For example, let W' be the apparent weight in water, and W the true weight.

Then $W-W'$ is the weight of water displaced.

Hence

$$\frac{W}{W-W'} = \frac{\text{weight of solid}}{\text{weight of same volume of water}}$$
$$= \text{specific gravity of solid.}$$

We have pointed out that when B is greater than W the body will rise. In order, then, to determine the specific gravity of such a body, it is necessary to keep it immersed by loading it with a solid of high specific gravity such as lead; this lead is called a *sinker*. Weigh the body in air (W); weigh the lead *in water* (W''), and the body and lead attached together in water (W').

Then, if the body could be weighed alone in water, its apparent weight would be $W'-W''$; this will be a negative quantity. Its "loss of weight"—*i.e.* difference between weight in air and apparent weight in water—is $W-(W'-W'')$, and this must be the weight of the water displaced by the body alone. Hence

$$\text{Specific gravity} = \frac{W}{W-W'+W''}.$$

Archimedes' principle may be illustrated as follows: A solid cylinder is provided, which fits so as exactly to fill a hollow cylinder, or can be hung attached to it below. If the two be "balanced" on a balance when in air, and then the solid be surrounded by water, the adjustment of the balance is upset; it can be restored by filling the hollow cylinder with water. Hence the apparent loss of weight of the solid in water is equal to the true weight of the same volume of water.

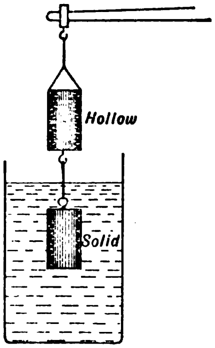


Fig. 140.

If the vessel of water be placed on a scale pan and the balance adjusted, and the solid be then suspended in the water from a support which is not in contact with the balance (Fig. 141), the scale pan descends as though the vessel of water had increased in weight. If the balance be readjusted the apparent increase in weight is exactly equal to the *buoyancy* of the solid. This is an interest-

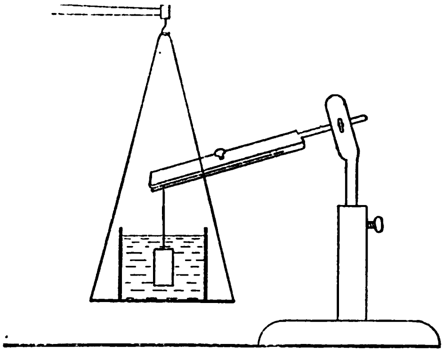


Fig. 141.

ing example of the law of equality of mutual actions (p. 35). If the liquid press up on the solid, this must at the same time press downward on the liquid with an equal force. Since weighings can be made with very great accuracy, this is probably one of the most accurate ways of demonstrating the equality of mutual action.

The mode in which the force is transmitted to the vessel may be described as follows: The presence of the solid in the liquid raises the free surface of the latter. Hence the base of the vessel is at a greater depth below the surface, and the force upon it is greater. If we suppose the vessel to be cylindrical and of cross-sectional area A, the water is raised a height h, where hA is the volume of the solid. The pressure on the base is increased by an amount swh (where s is unity); and the *force* on the base is increased by $wh \times A$ = weight of volume of water displaced. The student should prove that a similar result is true for a conical vessel, and, indeed, for a vessel of any shape; but he must remember to take into account the vertical component of the forces on the inclined surfaces.

FLOATING BODIES.

If a body has less specific gravity than the liquid, the buoyancy is greater than the weight of the body, and the latter can only be kept immersed by means of an additional force; *e.g.* a cork bobs up to the surface, if it is released under water. Such a body *floats*—*i.e.* rests so that part is under and part above the liquid. The fraction that is immersed is determined by the fact that the upward force due to the liquid (*i.e.* the buoyancy) must for equilibrium be equal to the weight of the body—*i.e.* B = W. Let V and s be the volume and specific gravity of the solid, v that part of the volume of the solid *at a lower level* than the free surface, and s_0 the specific gravity of the liquid. Then

$$wVs = wvs_0$$

$$\text{or } \frac{v}{V} = \frac{s}{s_0}.$$

In the above we have assumed that the buoyancy is the weight of a volume, v, of liquid, where v is defined as above. That this is a correct measure of it can be seen by the consideration of the part of the solid of volume v with a flat top held *just below* the free surface. Since it is totally immersed the buoyancy is s_0wv. The force due to the infinitesimal layer of water above the solid is itself vanishingly small, and no difference is made by removing it; the forces due to the liquid are then the same as for any floating body occupying the same region in the liquid.

This property affords a ready means of comparing the specific gravities of different liquids.

The device for doing so is called a hydrometer. This consists of a closed glass vessel having a narrow uniform stem, and loaded at its lower end with mercury or shot. It requires first to be graduated.

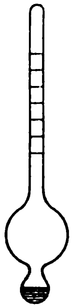


Fig. 142.

It is allowed to float in a standard liquid (sp. gr. = s_0), and a mark made on the stem at the level of the surface of the liquid. It is then placed in a second liquid of known specific gravity s', and the surface level again marked. If it be now placed in another liquid of unknown specific gravity, this can be found from the new position of the liquid surface. To read off this position, a scale must either be marked on the glass, or a paper scale can be inserted in the hollow stem before it is sealed up. If the scale is intended to indicate the specific gravity directly it must not be divided into equal parts: we now determine the necessary mode of graduation.

Let it be supposed to be made for use with liquids of higher density than water (sp. gr. = 1), the mark for water will then be near the top of the stem: let the volume of the hydrometer *up to this level* be V. Let the mark for the liquid of specific gravity s' be x' cm. below that for water, and let a be the cross-sectional area of the stem. The volume immersed in the case of the second liquid is $V-ax'$. Similarly let x be the number of centimetres below the top mark corresponding to the liquid of unknown specific gravity, s. Then

$$s_0 wV = s'w(V - ax') = sw(V - ax) = \text{weight of hydrometer,}$$

$$\text{or } s_0 = s'(1 - kx') = s(1 - kx)$$

where k stands for a/V, which is a constant.

$$\text{Hence } s' - s_0 = ks'x'$$

$$\text{and } s - s_0 = ksx;$$

$$\therefore \frac{x}{x'} = \frac{s - s_0}{s' - s_0} \cdot \frac{s'}{s}.$$

For example, let $s_0 = 1$, $s' = 1 \cdot 2$, and $x' = 4$ cm.; then the graduation corresponding to $s = 1 \cdot 1$ is

$$\frac{x}{4} = \frac{1 \cdot 1 - 1}{1 \cdot 2 - 1} \cdot \frac{1 \cdot 2}{1 \cdot 1} = \frac{6}{11};$$

that is, not halfway between the standard marks, but 6/11 of their distance apart.

The formula may be written

$$\frac{1}{s} = \frac{1}{s_0} + \frac{x}{x'}\left(\frac{1}{s'} - \frac{1}{s_0}\right).$$

Using the above data as a special case, this becomes

$$\frac{1}{s} = 1 + \frac{x}{4}\left(\frac{1}{1 \cdot 2} - 1\right) = 1 - \frac{x}{24},$$

$$\text{or } s = \frac{24}{24 - x}.$$

These formulæ show that equal increments in x correspond to equal decrements in *the reciprocal* of s.

For liquids lighter than water the weight requires to be adjusted, so that the mark for water is near the bottom of the stem.

Hydrometers such as the above are known as constant *weight* hydrometers. Besides these there are constant *volume* hydrometers. An open test tube with a mark near the top serves as a very simple one of this type. Sand is poured into the test tube until it just floats in water with the mark on the level of the surface. Test tube and sand are then weighed together; the weight, W_1, equals the weight of water displaced.

The test tube is then placed in the liquid of unknown specific gravity, and the sand adjusted till the mark is again level with the surface; both are weighed together once more. The weight, W_2 equals the weight of liquid displaced. Since the volumes displaced are the same in both cases, we have

$$\frac{W_2}{W_1} = \frac{\text{weight of a volume of liquid}}{\text{weight of equal volume of water}}$$
$$= \text{specific gravity of liquid.}$$

So far we have only found the fraction of the volume immersed when a solid floats on a liquid. We must now find the position of equilibrium. The weight of the body acts downwards through its own centre of gravity; the buoyancy, B—equal to the weight of liquid displaced—acts upwards *through the centre of gravity of the displaced liquid*; this latter point is called the centre *of buoyancy.* Now, for the body to be in equilibrium, these forces must have the same line of action; hence the centre of gravity of the solid must be vertically above or below the centre of buoyancy. Thus a homogeneous cube could be in equilibrium with two faces horizontal, or with four edges horizontal and the remainder at 45°.

In the case of a long cylinder both the vertical and horizontal positions are positions of equilibrium. But for one the equilibrium is stable, for the other unstable; the criterion of stability will now be found.

If the body is moved sideways or rotated about a vertical axis the equilibrium is not disturbed; for the volume immersed remains constant, and the two centres of gravity concerned are still one over the other; thus the equilibrium is neutral for such a displacement.

If the body is raised the buoyancy is decreased, hence when it is released it will move downwards in the direction of its own weight; similarly, if it is lowered and then released, it will rise, since the buoyancy is then the preponderating force. The equilibrium for such displacements is therefore necessarily stable.

We need only consider, then, the character of the equilibrium with regard to a small rotary displacement, about a horizontal axis, such that the volume immersed is constant.

Let the thick line represent the body in its equilibrium position, and the dotted line in a slightly displaced position, such as we have described. The line AOB on the body rotates

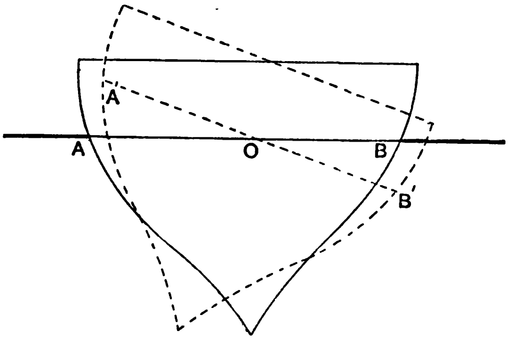


Fig. 143.

into the position A′OB′, round an axis of which O is the trace; and since the volume immersed is by supposition to remain the same, the wedge-shaped volume which has emerged from the fluid on the left, must be equal to the volume which has entered on the right.

Let each of these volumes be divided into vertical cylinders of very narrow cross-sectional area, the height of any one such cylinder at distance y from the axis O must be y tan AOA′, or with sufficient approximation since the angle is small, $y\theta$ when θ is the angle AOA′ expressed in circular measure. On account of the equality of the two wedge-shaped volumes,

$$\Sigma_1(ay\theta) = \Sigma_2(ay\theta)$$

where Σ_1 and Σ_2 denote summation on the left hand and on the right hand respectively of the axis.

Now θ cancels out, being the same for every term on both sides, and the remaining expressions denote the sum of the moments about the axis O, of the horizontal areas of which A′O and OB′ are the traces; at least, this is so if we attribute opposite signs to them. But the line for which this is true must pass through the centroid of the whole area whose trace is AB—*i.e.* the part of the area of the free surface of the fluid which is cut off by the floating body, or, as it is called, the *plane of flotation.* That this is so will be seen when it is recalled that the centroid of an area coincides with the centre of gravity of a homogeneous lamina sensibly coinciding with the area. In this volume we will only consider the case of bodies having a symmetric cross-section, and for which this centroid is vertically over the centre of gravity in the equilibrium position.

Having determined the axis round which the supposed rotation takes place, we proceed to find the couple which acts in any position of the body. Although by assumption the total buoyancy remains unchanged in amount by the rotation, it no longer acts through the same point in the body.

Suppose it acts through H′, then the couple is W (= B) multiplied into the perpendicular distance between B and W. This perpendicular distance is $\overline{GM} \sin \theta$ where M is the point of intersection of the new line of buoyancy and the old line HGM. M is called the *metacentre* and

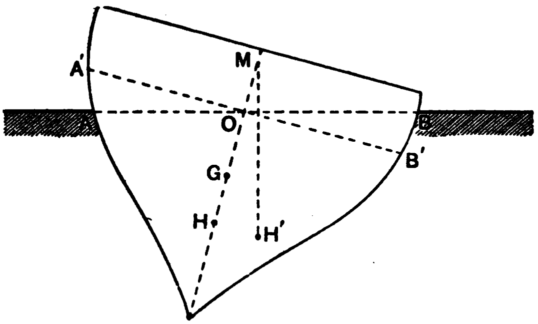


Fig. 144.

the question as to stability is decided by its position with respect to the centre of gravity of the body. For suppose M is above G, as shown on the figure (Fig. 144), the couple evidently acts counter-clockwise, and therefore tends to restore the body to its original position. On the other hand, if M is below G, then H′ must be to the left of G, and the couple is clockwise, and therefore acts so as to further increase the small rotation that the body has received. We may say, then, that the equilibrium is stable or unstable, according as the position in the body of the metacentre is above or below the centre of gravity, G. The length $\overline{GM}$ is called the *metacentric height.*

In order to find the metacentric height we may regard the total buoyancy as divided into three parts; one part equal to the weight W acting at the point H, which would be due to the body of liquid lying below the original plane of flotation A′OB′; and since this volume of liquid is the same as that displaced in the equilibrium position, the point H is the old centre of buoyancy simply rotated with the body; a second part due to a volume of liquid BOB′; and a third part equal to the weight *reversed in sign* of a volume of liquid AOA′. The total moment of the forces round O must include the moments of these, and also that of the weight of the body.

Consider first the volume BOB′ (Figs. 144, 145).

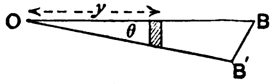


Fig. 145.

Each vertical cylinder of area a, into which it can be divided, has a weight given by $g\rho_0 a y\theta$ where y is its distance from O, and the moment of it about O is $g\rho_0 a y^2\theta$; so that the total moment of this volume of fluid is $g\rho_0\theta\Sigma_2(ay^2)$. Similarly for the left-hand wedge we have $g\rho_0\theta\Sigma_1(ay^2)$, and it must be remembered that this has to be reckoned positive, for a negative force on the left side of the axis has the same effect as a positive force on the right.

The moment of the old buoyancy acting at H is $-W \times \overline{HO} \sin\theta$, or, since θ is to be very small, $\overline{HO}.\theta$; and the moment of the weight acting through G is $W \times \overline{GO}.\theta$. Hence the total moment about O is

$$-W \times \overline{HO}.\theta + W.\overline{GO}.\theta + g\rho_0\theta.\Sigma_2(ay^2) + g\rho_0\theta\Sigma_1(ay^2).$$

This reduces to a counter-clockwise moment

$$[-W \times \overline{HG} + g\rho_0 Ak^2]\theta$$

where A is the total area of the plane of flotation, k is the spin-radius about an axis through O of a lamina sensibly coinciding with it, and $\overline{HG}$ is to be reckoned positive when G is above H. Hence, for stability

$$g\rho_0 Ak^2 > W \times \overline{HG}$$

or

$$\frac{Ak^2}{V} > \overline{HG}$$

where V is the volume of fluid displaced.

An alternative value of the moment is obtained in terms of the metacentre height $\overline{MG}$; it is evident from Fig. 144 that it is equal to $W \times \overline{MG}.\theta$, in which expression $\overline{MG}$ is to be reckoned positive if M is above G. Hence, equating these values, we get

$$W(\overline{HG} + \overline{MG}) = g\rho_0 Ak^2$$

or

$$\overline{HM} = \frac{g\rho_0 Ak^2}{W} = \frac{Ak^2}{V};$$

hence, for stability, $\overline{HM} > \overline{HG}$ or M must be higher than G.

For example, consider the stability for rotation about a longitudinal axis of a wedge-shaped body of density ρ and with its apical line lowest and horizontal. Let l be its length, b the breadth of its horizontal face, and h its vertical height. The volume immersed bears to the total volume the ratio ρ/ρ_0. Hence the depth h_0 of the apex below the plane of flotation is $h \times \sqrt{\rho/\rho_0}$. The breadth b_0 of the plane of flotation is $b \times \sqrt{\rho/\rho_0}$, and its area is this multiplied by l. The value of k^2 is $\frac{1}{12}b_0^2$.

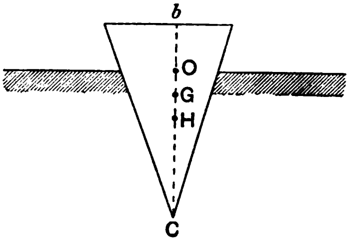


Fig. 146.

Hence (Fig. 145)

$$\overline{\text{HM}} = \frac{bl\sqrt{\rho/\rho_0} \times \frac{1}{12} \times b^2 \times \rho/\rho_0}{\rho/\rho_0 \times \frac{1}{2}bhl} = \frac{2}{12} \cdot \frac{b^2}{h} \cdot \sqrt{\frac{\rho}{\rho_0}}.$$

Again,

$$\overline{\text{CG}} = \tfrac{2}{3}h \text{ and } \overline{\text{CH}} = \tfrac{2}{3}h_0 = \frac{2h}{3}\sqrt{\frac{\rho}{\rho_0}}.$$

Hence

$$\overline{\text{HG}} = \tfrac{2}{3}h\left(1 - \sqrt{\frac{\rho}{\rho_0}}\right);$$

and ∴

$$\overline{\text{MG}} = \frac{2}{12}\frac{b^2}{h}\sqrt{\frac{\rho}{\rho_0}} - \tfrac{2}{3}h\left(1 - \sqrt{\frac{\rho}{\rho_0}}\right)$$

$$= \frac{b^2 + 4h^2}{6h} \cdot \sqrt{\frac{\rho}{\rho_0}} - \tfrac{2}{3}h.$$

The equilibrium is neutral when

$$\sqrt{\frac{\rho}{\rho_0}} = \frac{4h^2}{b^2 + 4h^2},$$

and this, of course, determines the border value between stability and instability. The condition for stability may be written

$$\frac{b^2}{4h^2} > \left(\frac{\rho_0}{\rho}\right)^{\frac{1}{2}} - 1.$$

The motion considered above is similar to the *rolling* of a ship; the above body may also *pitch* like a ship—*i.e.* rotate about a horizontal axis at right angles to its length. There will be a different metacentre for such motion, and its difference arises from the different value of the spin-radius about this second axis. The new value of k^2 is $\frac{1}{12}l^2$, so that

$$\overline{\mathrm{HM}'} = \frac{1}{6}\frac{l^2}{h}\sqrt{\frac{\rho}{\rho_0}}$$

and $$\overline{\mathrm{M'G}} = \frac{l^2 + 4h^2}{6h}\sqrt{\frac{\rho}{\rho_0}} - \frac{2}{3}h;$$

and the value of ρ/ρ_0, which gives neutral equilibrium, is

$$\left(\frac{4h^2}{l^2 + 4h^2}\right)^2.$$

A ship is much more stable for pitching than for rolling motion, owing to its length being many times its breadth.

In practice the metacentric height is determined by rolling a heavy weight—such as a big gun—across the deck, and observing the rotation produced by means of a plumb line. If the end of a plumbline of length L moves through s, the angle turned through is s/L with sufficient approximation. If b be the breadth of the deck, moving a weight, w, across it is equivalent to introducing a force of moment

$$w(b \cos \theta + \overline{\mathrm{OG}} \sin \theta)$$

about G when in the inclined position. This is balanced by the buoyancy, W, acting vertically upward through M, and whose moment about G is

$$\mathrm{W} \,.\, \overline{\mathrm{MG}} \sin \theta;$$

therefore

$$\overline{\mathrm{MG}} = \frac{w}{\mathrm{W} \sin \theta}(b \cos \theta + \overline{\mathrm{OG}} \sin \theta),$$

or, with sufficient approximation,

$$\overline{\mathrm{MG}} = \frac{wb}{\mathrm{W}\theta},$$

since the angle θ is always very small.

The conditions for stability given here are those arising from the consideration of the couple in a position near that of equilibrium in still water, and the stability may be defined as *static stability*. But other considerations must be introduced before it can be decided whether a ship will ride steadily in water which is in motion, as in the wave-covered ocean. These are beyond the scope of this book; but it may be mentioned that, if the time period of a wave is nearly that of the rolling motion of the ship, the latter may be set into dangerous motion even though its metacentre be high, and in such a case greater steadiness may be secured by *lowering* the metacentre. The greater steadiness arising from the adjustment of the *motion* of the ship may be distinguished as kinetic stability.

STATICS OF A GAS.

The peculiarity which distinguishes an ordinary gas from a liquid is its much greater change of volume with change in pressure and temperature.

The complications which depend upon the temperature belong to the subject of heat; in a mechanics manual, in order to avoid them, we will suppose the temperature to be maintained constant. Under this restriction the law of variation of volume with pressure is this: At constant temperature the product of the pressure and volume of a given mass of gas is nearly a constant.

The first experiments were made by Boyle with a tube, as shown in Fig. 146. Pouring mercury into the open limb, a certain mass of air becomes enclosed in the closed limb. The volume of the gas enclosed is proportional to the length of tube it occupies, while its pressure is in excess of that of the external atmosphere by that due to the column of mercury of height equal to the vertical distance between the two mercury surfaces. The length of this column can be varied by pouring in fresh quantities of mercury; the corresponding pressures and volumes are then observed.

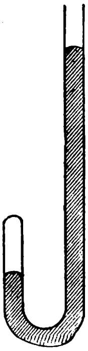


Fig. 146.

Boyle considered the product to be exactly constant; refined experiments show that, even for such gases as oxygen, hydrogen, nitrogen, etc. (the so-called "permanent" gases), at ordinary temperatures, the constancy is only approximate; and for gases that are easily condensable into liquids, such as carbon dioxide or sulphur dioxide, the law fails entirely near the liquefying point.

We will assume the law to be true in the succeeding pages.

The following theorems are true for gases as well as liquids, and the proof of them is effected in the same way in both cases:

1. The pressure at a point in a fluid at rest is independent of the aspect of the surface to which it relates.
2. The difference between the pressures at any two points is equal to the weight of a cylinder of the fluid of unit cross-sectional area, and of length equal to the vertical height between the two points.
3. Archimedes' principle.
4. Pascal's principle.

The pressure of a gas is supposed to arise in the main from its particles being in rapid motion. Each particle that comes against the side of the containing vessel has its motion reversed, and this change of motion implies a *force* between the particle and vessel. The average value of this force *per unit area* is the pressure on the vessel;

the pressure of the main body of the gas upon which its density depends is greater than this owing to attraction between the particles.

If a gas be admitted into the Torricellian vacuum of a barometer tube of unit cross-section area, the weight of gas enclosed gives, not the pressure of the gas, but merely the difference in pressure between its upper and lower surfaces.

In applying theorem 2 and 3, the density may be taken constant for any short column or small volume, because the weight of such a column or volume is very small; and in such cases the calculations are made just as for liquids, the density having first been ascertained at ANY point of the gas.

The following examples will illustrate the above statements:—

1. The volume of a gas being one cubic metre when the pressure is that due to 76 cm. of mercury, find the volume when the pressure is that of 19 cm. of mercury.

$$p_1 v_1 = p_0 v_0$$
$$19 \times v_1 = 76 \times 1$$
$$v_1 = 76/19 = 4 \text{ cubic metres.}$$

2. The density of air at 76 cm. pressure and 0°C. is ·001293 grms. per c.c.; find its density at a pressure of 19 cm. and the same temperature.

The density varies inversely as the volume; hence the answer is ·001293/4 = ·000323 grms./c.c.

3. A bubble of air occupies one cubic millimetre

at the bottom of a tank of water one metre deep; what will be its volume if it rise to the surface when the atmospheric pressure is 10^6 dynes per square cm.?

$$\begin{aligned} \text{Pressure at top} &= 10^6; \\ \text{pressure at bottom} &= 10^6 + 100 \times 981 \\ &= 1{\cdot}0981 \times 10^6. \end{aligned}$$

Hence volume at top = 1·0981 cubic millimetres.

4. An inverted test tube is sunk in water without letting any of the air within it escape. How deep must it be sunk in order that the air may only half fill it?

The pressure of the air inside is the same as the pressure at the inside surface of water. Hence this surface must be as much below the free surface as the effective surface is above it.

5. What is the apparent loss of weight of a 1000 grms. brass weight due to being placed in air at standard pressure (76 cm.) and 0°C.? Density of brass 8·4 grms./c.c., and of air ·0013 grms./c.c. nearly.

The volume of the brass is 1000/8·4 c.c. The apparent loss of weight is the weight of this volume of air, or $1000 \times {\cdot}0013 \div 8{\cdot}4$, or ·15 grms. weight.

6. The weight of a vertical column of air at standard pressure and temperature, one sq. cm. in cross-section and one metre high, is

$${\cdot}001293 \times 100 = {\cdot}1293 \text{ grms. weight}$$

or

$${\cdot}1293 \times 981 = 126{\cdot}8 \text{ dynes.}$$

The pressure of the atmosphere is about 10^6 dynes/cm.^{2}; hence the *difference in pressure* between the bottom and top of such a column is only about an eight-thousandth part of the pressure.

It follows that, if the atmosphere was of the same density at all levels, the height, in order to produce the actual pressure at the surface of the earth, would be about 8000 metres. [Taking $g = 981 \cdot 17$—the value at Greenwich—the height is 7988 metres.] This height is called "the height of a homogeneous atmosphere."

In the foregoing problems we have supposed the density to be the same throughout the volume of gas under consideration. But the change in pressure with change in level involves a proportionate change in density; and when long vertical columns of gas are dealt with, the error arising from the assumption of constant density becomes considerable. The homogeneous atmosphere is, for example, not even an approximation to the real height.

To illustrate this we will make the supposition that the temperature is constant throughout; we do this merely to indicate how the change of density is allowed for.

Divide the whole height of the atmosphere into equal lengths, as shown in Fig. 147, the heights of the several portions from the ground being 0, x, $2x$, $3x$, etc. If these lengths are small we may, without much error, consider the density

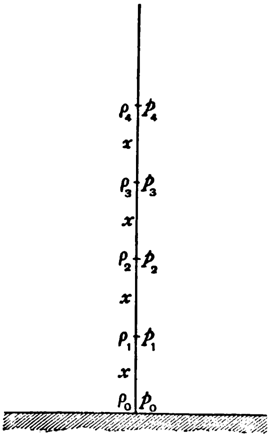


Fig. 147.

to be constant throughout any one, but to vary suddenly in passing from any one to the next. First suppose that the density is that corresponding to the bottom of each interval, and denote it by ρ_0, ρ_1, ρ_2, etc., the pressures being p_0, p_1, p_2, etc.

Remembering that the difference of pressure between any two points is equal to the weight of a unit cross-section cylinder of air extending between the two points, we have

$$\begin{aligned} p_1 &= p_0 - g\rho_0 x \\ p_2 &= p_1 - g\rho_1 x \\ p_3 &= p_2 - g\rho_2 x, \text{ etc.} \end{aligned}$$

But $\rho_1 = \rho_0 \frac{p_1}{p_0}$, $\rho_2 = \rho_0 \frac{p_2}{p_0}$, etc.

Hence putting $g\rho_0/p_0 = k$, we write

$$\begin{aligned} p_1 &= p_0 - p_0 kx = p_0(1 - kx) \\ p_2 &= p_1 - p_1 kx = p_1(1 - kx) \\ p_3 &= p_2 - p_2 kx = p_2(1 - kx), \text{ etc.,} \end{aligned}$$

$$\text{so that } p_n = p_0(1 - kx)^n;$$

i.e. the pressures at the equidistant points would be in a geometric progression of ratio $1 - kx$.

If we had supposed that in each length the density were that at the top of the particular length we would have obtained the following relations:

$$p_0 = p_1 + p_1kx = p_1(1 + kx)$$
$$p_1 = p_2 + p_2kx = p_2(1 + kx), \text{ etc.,}$$
$$\text{so that } p_0 = p_n(1 + kx)^n.$$

Hence, by one supposition, each pressure bears to the one *below* it the ratio $1 - kx$, whereas, according to the second relation, it bears to it the ratio $1/(1 + kx)$. The difference between these arises from the fact that an error is committed in taking the density as constant throughout each length. In the first case we took it at a value greater than its average value throughout the length, thus the ratio $1-kx$ is too small a fraction; in the second case we took it at a value less than its average, thus the ratio of the pressure is smaller than $1/(1 + kx)$. But when x is taken smaller and smaller, these ratios become more and more nearly equal, and their nth powers become more nearly equal, and the true value of the ratio of p_n/p_0, which lies between $(1 - kx)^n$ and $[1/(1 \times kx)]^n$, must be the value which these then become.

Let X be the distance between the two points, so that $X = nx$.

The two ratios are then $\left(1 - k\frac{X}{n}\right)^n$ and $\left(1 + k\frac{X}{n}\right)^{-n}$.

Expand by the binomial theorem,

$$\left(1 - k\frac{X}{n}\right)^n = 1 - nk\frac{X}{n} + \frac{n(n-1)}{\underline{|2}}\left(k\frac{X}{n}\right)^2 -$$

$$\left(1 + k\frac{X}{n}\right)^{-n} = 1 - nk\frac{X}{n} + \frac{-n(-n-1)}{\underline{|2}}\left(k\frac{X}{n}\right)^2 -$$

and when *n* is infinite each of these becomes

$$1 - kX + \frac{k^2X^2}{\underline{|2}} - \frac{k^3X^3}{\underline{|3}} + \text{etc.}$$

This is therefore the true ratio that we seek; it can be written ϵ^{-kX} where ϵ is the base of *natural* logarithms and is the incommensurable number 2·718281828. . . . . . . Since X occurs in the index of ϵ it follows that p varies in geometric progression for equal steps in X; but the ratio is not the approximate one which we first found.

It follows also that an infinite height would need to be reached before the pressure became zero. It is of interest to observe that on another simple hypothesis (which supposes the temperature, instead of being constant, to vary according to the adiabatic law) the pressure would become zero at a height less than twenty miles. Judging from observations of meteors, which only become visible when they are made incandescent by friction against the air, the true height is over two hundred miles.

EXAMPLES TO CHAPTER XIX.

1. Calculate the difference of pressure between two points one mile apart, and on the same vertical in sea water: sp. gr. = 1·026. Express the result in (i) dynes per cm^2; (ii) lb. weight

per sq. inch; (iii) lb. weight per sq. ft.; (iv) tons weight per sq. inch.

2. Assuming that the pressure at the higher point is 14·47 lb. weight per sq. inch, calculate the pressure at the lower point.

3. Calculate the magnitude of the force on each of the faces of a cube (length of side = 2 ft.) which is filled with mercury of sp. gravity 13·6.

4. Calculate the pressure at a depth of 4 cm. of mercury, the pressure at the free surface being 10^6 dynes per sq. cm.

5. Calculate the difference in level of the free and effective surfaces, when the atmospheric pressure is equal to that of 76 cm. of mercury, in the case of the following liquids: (i) mercury; (ii) pure water; (iii) sea-water.

6. A barometer tube, instead of being attached to the same stand as the vessel of mercury into which it dips, is suspended from one arm of a balance. What is the apparent weight of the glass tube, and how does it vary as the "barometric height" fluctuates?

7. Calculate (i) in dynes per sq. cm., (ii) grams weight per sq. cm., (iii) lb. weight per sq. inch, the pressure equivalent to a standard atmosphere—*i.e.* to a barometric height of 76 cm. of mercury.

8. A cistern with rectangular sides is kept full of water. At which point in the side should a strut be placed, if additional support is necessary?

9. Calculate in magnitude and position the

resultant force on a lock gate 12 ft. wide, the water being 10 ft. deep on one side and 20 ft. on the other.

10. Show how to divide by a horizontal line a plane rectangular area, which is totally immersed in a fluid, in such a way that the forces on the two parts shall be equal, two sides of the rectangle being horizontal: (i) when the pressure at points on the upper edge is zero; (ii) when the pressure is p_0.

11. A cube is filled with a liquid and held with a diagonal vertical. Find the magnitude of the force on each face.

12. How deep below the effective surface must a square area with two sides horizontal be sunk in water, in order that the centre of pressure may be one-sixth the length of the side below the centre?

13. Find the depth of the centre of pressure of a trapezium with two sides horizontal. (Divide it into two triangles.)

14. A cylindrical jar is filled with water. Find the resultant force on the half of the curved surface cut off by a vertical plane passing through its axis.

15. A hemispherical bowl, with rim uppermost and horizontal, is filled with water. Find the resultant force (i) on the bowl, (ii) on the half of it cut off by a vertical plane through its centre.

16. A cylinder closed at both ends is held with its axis horizontal, and is filled with fluid. Find

the resultant force on the half of the curved surface cut off by a vertical plane through its axis.

17. A hemispherical bowl, whose external radius is 4 in. and whose mass is 2 lb., floats in water. What volume of lead of sp. gr. 11·35, must be placed in it to sink its rim level with the water?

18. A cylinder of iron floats with its axis vertical on the surface of a vessel of mercury. If water is poured on the top of the mercury so as to cover the iron, through what fraction of its height will the cylinder rise? (Sp. gr. mercury 13·6, iron 7·8.)

19. A long cylinder of wood (sp. gr. ·65), loaded at the bottom, floats with its length vertical in water, and 20 cm. of it is above the surface. If oil (sp. gr. ·75) to a depth of 1 cm. be poured on the water, what length of wood will be above the oil?

20. A triangular wedge, the sides of whose cross-section are 6, 10, and 10, floats in water. What is its density if the equilibrium for rolling motion is neutral when the apical line is lowermost?

21. Show that the metacentre of a floating sphere coincides with its centre of gravity.

22. Find the metacentric height for a solid cone floating apex downwards, and examine the stability.

23. A liquid occupies the lower part of a U-tube; another liquid of less specific gravity

is poured gently into one limb. If the liquids do not mix, show that the level in the other limb will rise, but not so high as the upper surface of liquid in the first limb.

24. One liquid lies over another in a vessel, and a vertical cylinder floats in equilibrium in these liquids, being partly in one, partly in the other. Show that if it protrudes above the upper liquid a different equation is necessary for calculating the equilibrium position from that which is necessary if the upper end of the cylinder is in the upper liquid.

CHAPTER XX

UNITS AND DIMENSIONS

THE complete expression of any physical quantity consists of two parts; one of these is the name of the *unit* employed, and the other is a *number* specifying the ratio to the unit of the quantity in question. Thus a length is expressed as we have seen (page 5), as a certain number of times the length of a standard metre (or centimetre) or yard.

The particular length chosen as a unit is quite arbitrary, and any quantity whatever might be measured in terms of an equally arbitrarily selected unit. But greater simplicity arises by only selecting arbitrarily as few as possible. Thus, having fixed the units of length and time, the unit of velocity is made to depend upon them; unit velocity is the rate which will carry a body unit distance in unit time. Here, we express a relation between a certain length and a certain time; and the unit required in terms of which to measure a velocity is made to depend upon this relation.

Now the main part of physical science consists in finding the relations which exist between quantities of various kinds. Names, such as velocity, acceleration, etc., are given to certain of these relations, and in each case the relation itself provides the means of fixing the unit required in measurements.

A unit based upon any such relation is called a *derived unit*; one which is chosen independently of other quantities is called a *fundamental* unit.

In Mechanics it has been necessary to introduce only three fundamental quantities, *viz.* LENGTH, TIME, MASS, each of these being *sui generis.*

We will give as simple an illustration as possible of the above statements. Our physical notion of an area is that which has length in two directions. Fix attention on a rectangle which is a feet long and b feet broad where a and b are integers. If lines, one foot apart, be ruled parallel both to a and b, the rectangle is divided thereby into smaller rectangles, each of which is *called a square foot.* The number of these smaller rectangles is ab; *i.e.* the number which specifies the magnitude of the rectangle in terms of a small one is obtained by multiplying the lengths of the two perpendicular sides. But we might have proceeded differently. Any other area whatever might have been chosen as a unit; but, if so, the number specifying the magnitude of the large rectangle will no longer be equal to the product of the lengths of the sides, each expressed in feet. In other words, to become the value required this product would require multiplying by a certain factor. For example, suppose that a rectangle ½ foot long by 1 foot broad had been selected as a unit, then, clearly there will be twice as many of these areas as before in the large rectangle, and therefore the product referred to must be multiplied by two. Now it is clear that this choice of a new unit for *areas* which is not based on the units of length already fixed leads to greater complication; this is well illustrated by a number of our British units. Thus a *volume* can be expressed in terms of three lengths multiplied together, if as unit of volume be chosen a cube each side of which is a unit long. But besides this we have a *gallon* as a unit; and since there are 6·232 gallons in a cubic foot, it follows that the number of gallons in a given rectangular volume is equal to 6·232 times the number obtained by multiplying together its length, breadth, and depth, each expressed in feet. The tendency in scientific work is to avoid this unnecessary complication; or, at least, to provide that the factor, if present, shall be a multiple of ten. Thus we have, as unit of volume, a cube whose side is 1 centimetre, that is, a *cubic centimetre*; but we have also another unit, *viz.* the *litre*, which is equal to 1,000 cubic centimetres. The presence of a multiple of ten has so little inconvenience that it is tolerated whenever the litre is more nearly comparable with the volume that is to be compared with it; very big numbers are avoided thereby.

A precisely similar method is adopted in fixing the numerous other units which we have had to introduce. Thus, instead of selecting an arbitrary velocity as unit, we choose one foot per second, or one centimetre per second, or one mile per hour; or, in general, the velocity which a body has which travels one unit length in each unit time.

Dimensions of Units. Each unit, we have said, is made to depend upon some or all of three fundamental units; we require a concise way of expressing its mode of dependence. Thus, the unit area is obtained by multiplying two equal lengths together; it is obvious then that if we double the unit length the unit area becomes *four* times as large. We indicate this fact, that the *change-ratio* of the unit area depends upon the change-ratio of the unit length, by saying that an area is of the same *dimensions* as the square of a length, or is of the second dimension as regards length, and we write it thus, in square brackets—

$$[\text{Area}] = [L^2].$$

Similarly a volume is of the same dimensions as the cube of a length, or

$$[\text{Volume}] = [L^3];$$

for three lengths are multiplied together in obtaining the specification of any volume. Again, a velocity is always measured by dividing a length by a time; we say, therefore, that its dimensions are unity as regards *length* and minus unity as regards time, or

$$[\text{Velocity}] = [LT^{-1}].$$

We shall now briefly deduce the dimensions of all the other quantities dealt with in this book.

Acceleration. Obtained by dividing a velocity by a time, or a length by two times:

$$\therefore [\text{Acceleration}] = [LT^{-2}].$$

Force. Obtained by multiplying an acceleration by a mass:

$$\therefore [\text{Force}] = [MLT^{-2}].$$

Momentum. Obtained by multiplying a velocity by a mass:

$$\therefore [\text{Momentum}] = [MLT^{-1}].$$

Impulse. Same as Momentum, the change of which it measures.

Energy and Work. Obtained by multiplying a force into a distance:

$$\therefore [\text{Energy}] = [\text{Work}] = [ML^2T^{-2}].$$

Kinetic energy is measured by multiplying a mass into the square of a velocity:

$$\therefore [\text{Energy}] = [ML^2T^{-2}], \text{ as before.}$$

Power or Activity. Obtained by dividing work by time:

$$\therefore [\text{Power}] = [ML^2T^{-3}].$$

Moment of a Force. Obtained by multiplying a force into a distance:

$$\therefore [\text{Moment}] = [ML^2T^{-2}],$$

that is, the dimensions are those of energy or work.

Moment of Inertia. Obtained by multiplying a mass into the square of a length:

$$\therefore [\text{M. of Inertia}] = [ML^2].$$

Angle, or its Sine, Cosine, or Tangent. Obtained by dividing a length by a length; each is, therefore, of zero dimensions.

The student may need to be warned that he must carefully distinguish this use of the term dimensions from its popular one, as when we speak of the dimensions of a room. According to the latter usage, it means merely the *magnitudes* of the lengths of the three sides. The meaning given above to the term has no reference to the *magnitudes* of the quantities referred to, but to the mode in which the fundamental units are involved in the derived ones. This conception was first introduced by Fourier.

Uses of the Conception of Dimensions. It has already been indicated that a knowledge of the dimensions of any physicial quantity facilitates the change from one unit to another. For example, since a product of two lengths is always involved in specifying an area, doubling the unit of length will mean quadrupling the unit of area;

the numerical magnitude of a given area when expressed in terms of the new unit will only be one quarter as great as when expressed in terms of the old one. In general, if the unit of mass is changed to one *a* times as large, the unit of length to one *b* times as large, and the unit of time to one *c* times as large, and if the dimensions of any physical quantity are $[M^xL^yT^z]$, then the unit of this quantity becomes $a^xb^yc^z$ times as large by the change; and the numerical magnitude of any quantity expressed in terms of the new unit is equal to the old magnitude multiplied by the *reciprocal* of $a^xb^yc^z$.

Problem. A certain momentum being given by the number 16 when pounds, feet, and seconds are the fundamental units, what number will express it when grams, centimetres, and seconds are the fundamental units?

$$\text{Here} \quad a = \frac{1}{453.6} \qquad x = 1$$

$$b = \frac{1}{30.48} \qquad y = 1$$

$$c = 1 \qquad z = -1$$

$$\therefore \frac{\text{New unit}}{\text{Old unit}} = \left(\frac{1}{453.6}\right)^1 \times \left(\frac{1}{30.48}\right)^1 \times 1^{-1}$$

$$= \frac{1}{453.6 \times 30.48} = \frac{1}{13825}.$$

Thus the new unit of momentum is very much smaller than the old one; the number of the C.G.S. units in the given momentum will be proportionately *large*—*i.e.* 16 × 13825 = 221200.

Another use to which a knowledge of dimensions is put is in checking formulæ. Suppose that a formula obtained for the time of swing of a pendulum was found to be

$$T = \pi \sqrt{\frac{l^2}{g}};$$

how can we test whether it is right? The dimensions of the quantity on the right side must be the same as that on the left, *viz.* [T].

Now on the right we have

$$\left[\left(\frac{L^2}{LT^{-2}}\right)^{\frac{1}{2}}\right] = [L^{\frac{1}{2}}T],$$

which is not [T]. The formula is, therefore, erroneous. This mode of checking, it should be observed, tells one nothing about the purely numerical factor π which is also wrong.

Again, suppose that we obtain the expression,

$$aX^2 + 2hXY + bY^2 = 0,$$

where X and Y are lengths, and a, h, b are purely numerical constants. The test of dimensions shows that this is a possible equation, for each term has the same dimensions, *i.e.* each is a quantity of the same kind, and it is therefore legitimate for them to be added together.

But the expression,

$$aX^3 + 2hXY + bY = 0,$$

where the letters have the same meaning as before, cannot represent any physical fact; for each term has distinct dimensions, and it is as meaningless to add their numerical magnitudes together as to add a volume and a length.

A third use to which dimensions are put is known as the *method of dimensions.* By means of it we can sometimes determine the shape of a formula without knowing how to calculate the formula itself. We will take a known formula to illustrate this method; let it be again the formula for the time of swing of a simple pendulum. If we can assume that it depends upon some power of the three independent quantities; the length of the pendulum, l; the mass of the bob, m; and the earth's force, W, each of which can vary without any change in the other two; and upon no other physical magnitude; then we may write

$$T \propto m^x l^y W^z;$$

and this expression must have the same dimensions on both sides. Now W being a force has the dimensions MLT^{-2}; therefore

$$\begin{aligned}[T] &= [M^x L^y (MLT^{-2})^z] \\ &= [M^{x+z} L^{y+z} T^{-2z}].\end{aligned}$$

The power of the mass on the left is zero; it must be zero on the right also, or

$$x + z = 0.$$

Similarly, by considering the length and time

$$\begin{aligned} y + z &= 0 \\ 1 + 2z &= 0.\end{aligned}$$

The last equation gives $z = -\frac{1}{2}$;

whence $x = +\frac{1}{2}$ and $y = +\frac{1}{2}$.

Thus—

T must vary as $m^{\frac{1}{2}} l^{\frac{1}{2}} W^{-\frac{1}{2}}$

$$\text{or } T \propto \sqrt{\frac{ml}{W}}.$$

$$\text{But } W = mg \therefore T \propto \sqrt{\frac{l}{g}}.$$

Hence, with the exception of a constant multiplier, the desired formula is obtained.

We might equally well have taken the length, the mass of the bob, and the acceleration due to gravity as being the independent quantities upon which the time period must depend: the same result would be obtained.

By means of the method of dimensions the form of many expressions has been determined, in cases where the complete solution of the problems has proved intractable to all known mathematical methods.

Of course, in applying this method, care must be taken to include every quantity upon which the desired quantity can depend.

Without making a complete investigation on these lines, we can often obtain useful hints towards the simplification of a problem. Thus, in the above case, since no mass occurs on the left-hand side, we know that if a mass occurs on the right in the numerator, then some mass must occur *to the same power* in the denominator.

Now, in the case of a simple pendulum, there is only one mass to refer to at all, *viz.* the mass of the bob; and since it is bound to occur to the same degree in both numerator and denominator, its influence will cancel out as regards numerical value as well. In other words, the time of swing of a simple pendulum *is independent of the mass of the bob*; and we may consequently at once put it equal to unity before carrying the calculation further.

Again, we conclude in the same way that the equilibrium position of a ladder resting against a rough wall must be independent of the mass of the ladder, because an angle has no dimensions, and consequently, if in the expression for it the mass of the ladder enters in the numerator, it must also enter to an equal degree in the denominator, there being no other mass involved.

MISCELLANEOUS EXAMPLES

1. Express 45 miles per hour in feet per second and in yards per minute.

2. One point is moving with a speed of 1 cm. per sec., another with a speed of 1 mile per hour; compare their speeds.

3. A point moving in a circle of 40 ft. radius makes $4\frac{1}{2}$ revolutions in 20 sec. Find its mean speed.

4. Two trains pass one another in opposite directions with velocities of 45 and 60 miles per hour respectively. Their lengths are 420 ft. and 350 ft. respectively. How long will they take to clear one another?

5. A train which is uniformly accelerated starts from rest and at the end of 3 sec. has a speed with which it would travel through 1 mile in the next 5 min.; find its acceleration.

6. Two trains set out from the same station along parallel lines; the one goes 5 miles northward, the other one 6 miles southward. Find the displacement of the latter with respect to the former.

7. Two trains run, the one N.E. a distance of 20 miles, the other S.E. through an equal distance. Find the displacement of the former relative to the latter.

8. Express the acceleration due to gravity in yard minute units ($g = 32{\cdot}2$ ft. per sec.^{2}).

9. A point moves from rest with a constant acceleration and passes over 36 yd. in 3 sec. What is its acceleration and also its speed after the 3 sec.?

10. A point moves upwards with an initial speed of 64 ft./sec. under a uniform acceleration of -32 ft./sec.

Where is it at the end of the first, second, third, fourth, and fifth seconds respectively?

11. Make velocity diagrams representing (1) the case of a point moving vertically with an initial velocity of 160 ft. per sec., and uniform acceleration of -32 ft. per sec. per sec.; (2) the case of a point moving downwards from rest under the same acceleration.

12. A stone is thrown vertically upwards with a speed of 320 ft. per sec. Five seconds afterwards another one is thrown vertically upwards from the same point and with the same speed. Where will they meet?

13. A ship is sailing due N. with a velocity of 10 knots, while another is steaming S.W. at the rate of 15 knots. Find the magnitude of the velocity of the second with regard to the first.

14. A ship is steaming at the rate of 8 miles an hour due north. A man runs due east right across her deck with a velocity of 8 miles an hour. What is his velocity with respect to the earth?

15. A circus rider is moving with the uniform speed of 1 mile in 2 min. 40 sec. round a ring of 100 ft. radius. Find his acceleration towards the centre of the ring.

16. A point moving in a circle of 10 ft. radius has a central acceleration of 10 ft. per sec. per sec. Find its speed.

17. A wheel of a carriage which is travelling at the rate of 7 miles per hour is 3 ft. in diameter. Find the angular velocity of any point of the wheel about the axle.

18. Two arrows whose masses are in the ratio of 1 to 2 are shot vertically upwards with equal impulses. Compare the times taken in their respective journeys till they arrive again at their starting point.

19. Masses of 3 and 2 oz. are hung from the right and left ends of a string in an Atwood's machine. If weights of 4 and 1 oz. are hung below the right and left weights respectively, calculate the tension in each string when the whole is moving freely.

20. By the Law of Equality of Action and Reaction the force with which a cart pulls a horse backwards is exactly equal at each moment to the force with which the horse pulls the cart forwards. Explain how this statement is consistent with their motion.

21. Discuss, with regard to the same law, the possibility of winning in a game of tug-of-war.

22. The weights in an Atwood's machine are 9 and 11 oz. respectively. They have moved for 5 sec. from rest when the string breaks. In what direction will the small weight afterwards move, and how far will it move before coming to rest?

23. A stone weighing 1 lb. is fastened to the end of a string 1 yd. long. How fast could you swing it round in a horizontal plane without the string breaking, if this can stand a stress of only 2 lb. weight?

24. If the stone were swung round in a vertical plane in which position would it be most likely to break, and why?

25. What is the apparent weight of a stone of mass 10 lb. situated on the equator, the speed of a point on the equator being 465 metres per sec.?

26. Two electrified pithballs, hung by two silk threads from one point, repel one another so as to make an angle of 60° between them. If the pithballs each weigh 1 gram, find the force between them?

27. A mass of 100 lb. moving on a horizontal surface with a speed of 10 ft. per sec. comes to rest in 2 sec. Find the coefficient of friction.

28. A mass of 100 lb. is moved along a horizontal plane by a constant horizontal force of 20 lb. weight. Determine the displacement in 10 sec., the coefficient of kinetic friction being ·17.

29. A force acting parallel to a plane and equal to a weight of 28 lb. is required to just draw a mass of 30 lb. up a plane inclined at 30° to the horizon. Find (*a*) the coefficient of friction and the total friction, (*b*) the force that would be necessary if the inclination were 45°.

30. A shot of 700 lb. moving with the velocity of 1200 ft. per sec. enters the side of a ship weighing 6000 tons and remains imbedded in it. Find the velocity it communicates to the ship. What is the loss of kinetic energy due to the impact?

31. A uniform hemisphere rests with its curved surface in contact with a rough horizontal table. It is gradually tilted by applying a horizontal force at a point on the circumference of its circular base. Prove that if $16\mu < 3$ the hemisphere will begin to slide instead of rolling when its base makes with the horizon an angle

$$\sin^{-1}\left(\frac{8\mu}{3 - 8\mu}\right).$$

32. A bullet of mass m is fired horizontally with velocity v into a block of mass M resting on a smooth table. Assuming that the block does not rotate, and that the penetration of the bullet is resisted by a constant force R, prove that when the bullet comes to rest in the block the latter will have moved through a distance

$$\frac{\mathrm{M}m^2v^2}{2(\mathrm{M} + m)^2\mathrm{R}}.$$

33. Given that 50 horse-power is required to drive a steamer 100 feet long at a speed of 8 knots, prove that 6400 horse-power will be required to drive a similar steamer, 400 feet long, similarly immersed, at 16 knots, assuming that the resistance is proportional to the wetted surface and the square of the velocity through the water.

Prove that in a given voyage from one port to another, the cost of coal per ton of cargo will be the same in the two steamers.

34. Determine in tons the greatest train an engine capable of exerting a uniform pull of 3 tons can take on the level from one station to the next, a mile off, in 4 min., supposing the resistance of the road is estimated at 20 lb. a ton, and the brake power at 400 lb. a ton in addition.

35. Eight equal and similar rods are connected by smooth joints so as to form the skeleton of a pyramid with a square base. If this framework is placed on a smooth horizontal table find the stress in each bar.

36. Two equal and similar cylinders float in different liquids, and make vertical oscillations in times t and t^1—respectively; compare the densities of the liquids.

37. The coupling between two railway carriages consists of a rod with a right-handed and a left-handed screw cut at its opposite ends and turning in nuts attached to the carriages. The pitch of each screw is ·25 in., and the rod is turned by a force of 56 lb. acting at best advantage at the end of a lever 15 in. long.

Find the force by which the carriages are drawn together.

38. A weight of 100 lb. is suspended from C, being supported by two rods, AC, BC, 6 and 8 ft. long, freely jointed together, whose other ends A and B rest on a frictionless horizontal plane and are kept from slipping by a cord AB, 10 ft. long, connecting them: the plane ACB is vertical. Find the thrust in each rod, and the tension in the cord arising from this load; also the fractions of the load sustained by the plane at A and B respectively.

39. Forces act upon a solid body, represented in position and magnitude by AB, twice BC, CA. Find the position and magnitude of their resultant.

Find the resultant of forces represented by AB, BD, DC, CA, where ABCD is a parallelogram, of which AC, BD are the diagonals.

40. A nail weighing $\frac{1}{2}$ oz. is driven 1 in. into a board by the fall from a height of 4 ft. of a hammer weighing 2 lb. Find the velocity of the hammer just before and just after impact, the loss of energy in impact, the average opposing force in the wood, and the duration of the event.

41. Prove that when two bodies collide the motion of their common centre of gravity is unaltered by the collision.

42. Prove that if the rope and pulley of a simple pulley

are supposed so stiff that a weight can lift only two-thirds of its own weight then the acceleration will be

$$\frac{\frac{2}{3}\,\mathrm{P}-\mathrm{Q}}{\frac{2}{3}\,\mathrm{P}-\mathrm{Q}}g$$

when P and Q are weights hung over the pulley.

43. The centre of gravity of the beam of a balance of which the weight is W, is at a distance c below its axis of suspension, and the axes of suspension of the scale-pans are at a distance d below the same axis, when the beam is in equilibrium in a horizontal position; prove that, when the scale-pans are removed, a weight w hung on to one of these axes will make the beam incline to an angle whose co-tangent is $(\mathrm{W}c+wd)/wa$, where $2a$ is the distance between the two axes.

44. A bar is in equilibrium under the action of forces applied only at its ends; prove that the resultant of the set that is applied at either end acts along the bar.

A light chain of length l, whose weight is neglected, has its extremities fastened to the two ends of a bar of length a, which is supported horizontally: prove that, when a heavy weight W is slung on the chain it produces a thrust in the bar of magnitude $\frac{1}{2}\,\mathrm{W}a\,(l^2-a^2)^{-\frac{1}{2}}$.

45. Prove that, assuming gravity to act in parallel lines, the weight of a rigid body always acts through the same point, however the body be turned about.

Prove that the centre of gravity of a thin hemispherical shell, closed by a plane base, and all made of uniform sheet metal, is at a distance from its base equal to one-third of its radius.

If the inside of the shell is filled with water, and when suspended from a point on the rim of the base, it hangs with the base inclined at an angle α to the vertical, show that the ratio of the weight of the water to that of the shell is

$$\frac{\tan\alpha-\frac{1}{3}}{\frac{3}{8}-\tan\alpha}.$$

46. Investigate a formula for the time required by a stone projected upwards with given velocity to attain a given height, the resistance of the air being neglected.

A stone is projected upwards from a catapult at a point B with a velocity of 80 ft. per sec., and 2 sec. later another is projected from a point A in the same straight line with the observer O and B, and with the same velocity. The observer sees them in the same line with his eye after 2 more seconds: deduce the ratio of the distances OA and OB.

47. Give the exact equation, supplied by the laws of motion, that connects the initial and final velocities of two balls, which, moving in the same straight line, collide and rebound.

A mass of 8 lb. moving north at a speed of 10 ft. per sec. is struck by a mass of 6 lb. moving east at 14 ft. per sec., and its direction of motion is thereby deflected through 30°, while its speed is increased by 1 ft. per sec.; show that the velocity of the other is diminished by 7·3 per sec., approximately, and find its new direction of motion.

48. Weights of 5 lb. and 3 lb. are suspended by a long fine cord over a frictionless pulley, and are allowed to move freely. Prove that, when the weight of the cord and pulley is negligible, the pulley supports the weight of $7\frac{1}{2}$ lb. while the motion lasts.

49. In a system of pulleys with one fixed and one movable block, in which the cord is attached to the axis of the movable block, then passes over the fixed one, then under the movable one, and then over the fixed one, find the weight P which, when attached to the cord, will support a given weight W hung from the movable block. (The blocks are so small that all the straight portions of the cord may be considered parallel.)

Show that, if the weights do not balance, the downward acceleration of W, when let go, will be

$$\frac{W - 3P}{W + 9P}g,$$

the weight of the cord being neglected, and that of the movable block being included in W.

50. A uniform quadrilateral plate balances in any position about an axis coincident with either diagonal. Prove that the quadrilateral must be a parallelogram.

51. Prove that, if the arms of a balance are unequal, the true weight of a body is equal to the geometric mean of the apparent weights, when it is weighed, first in one scale-pan and then in the other. Has inequality in the weights of the pans themselves any influence?

52. Prove that, in uniformly accelerated motion, the average velocity in any interval of time is equal to the arithmetic mean of the velocities at the beginning and end of the interval.

A constant force acts upwards on a particle initially at rest, and ceases when the particle has ascended through a height h, and the particle comes to rest after ascending through a further height h'. Prove (if possible without using formulæ) that the times of describing h, h' are directly as these heights, and that the ratio of the force to the weight of the particle is $(h + h') \div h$.

53. A mass of 1 ton falls from a height of 16 ft. on the end of a vertical pile, and drives it half-an-inch deeper into the ground. Assuming the driving force of the mass on the pile to be constant while it lasts, find (roughly) its amount in gravitation measure, and the duration of its action.

54. A uniform log of timber is carried by five persons, of whom one takes hold of the end of the log, and the others take hold of the ends of two cross-bars on which the log rests. Prove that the bars may be arranged in various ways so that the load shall be equally distributed between the bearers; also that of these ways the one in which the middle point of support is at greatest distance from the nearest of the other two is that in which the three points are at equal intervals of $\frac{5}{12}$ of the length of the log.

55. A horizontal uniform beam 12 ft. long and weighing

56 lb. rests on and is fastened to two props 5 ft. apart, one of which is 3 ft. from one end of the beam. A load of 35 lb. is placed (*a*) in the middle of the beam, (*b*) at the end nearest a prop, (*c*) at the further end. Calculate the weight that each prop has to bear in each case.

56. A uniform rod AB is movable in a vertical plane about a hinge at A. To B a string is attached which passes over a pulley at C vertically above A and such that AC = AB. From the free end of the string hangs a weight equal to half the weight of the rod. Find the position of equilibrium and the reaction of the hinge.

57. A body of known mass is placed so as to be capable of rotation about a fixed horizontal axis. Describe some method of finding the distance between the centre of gravity of the body and the axis.

58. Prove that if a body of any shape suspended at the end of a rope is lowered gently on to a smooth horizontal plane the rope will remain vertical. Determine the reaction of the plane in any position.

59. Given two forces, one in magnitude and direction, the other in direction only. Prove that the *least* equilibratory force is perpendicular to the second one.

60. Prove that a train can ascend a slope of 1 in $n/\tan\theta$, if θ is the limiting slope on which the train can be held with all wheels braked, and if the weight of the whole train is n times the weight on the driving wheels of the engine.

61. Determine the average thrust on the pedals required to propel a bicycle and rider weighing 200 lb. up an incline of 1 in 50, the cranks being 7 in. long, and the bicycle geared up to 70 in. (*i.e.* so that one revolution of the pedals makes the machine advance the length of the circumference of a wheel of diameter 70 in.).

62. If a wheel rolls without slipping on the ground and the speed of the centre is constant, what is the acceleration, in magnitude and direction, of any point on the rim; and, in particular, what is the acceleration of the particle in instantaneous contact with the ground?

63. Prove that when a sphere rolls down a rough plane, the coefficient of friction being μ, slipping will not take place unless $\tan\theta > 7\mu/2$ where θ is the inclination of the plane.

64. Two pairs of equal bars, AD and BC, AC and BD, are jointed together at the ends so that AC and BD are the diagonals of the quadrilateral ABCD; a point K is marked on the bar AD, a point L on AC, and a point M on BC, so that the three points lie on a line parallel to AB. Show that KL. LM remains constant for all the shapes that this jointed frame can assume.

Hence show that if L is fixed, and K describes a circle, then M describes another circle; deduce a solution of the problem of converting circular motion into rectilinear motion by linkages alone.

65. AB, BC, and CD are equal rods jointed at B and C. A and D rest on a smooth horizontal plane and B and C are joined to the middle points of CD and AB by equal strings of such a length that the angles at B and C are each 120°. Show that the action at B on BC makes an angle $\tan^{-1}\left(\frac{10}{\sqrt{3}}\right)$ with the vertical.

66. A circular ring of radius a and weight W is suspended in a horizontal position by three vertical threads, each of length l, attached to points on its circumference. Prove that, in order to hold the ring horizontal and twisted through an angle θ, a horizontal couple must be applied equal to

$$W\frac{a^2\sin\theta}{(l^2-4a^2\sin^2\theta/2)^{\frac{1}{2}}}.$$

67. A uniform rod AC, 15 in. long, is suspended by means of a string 21 in. long, which is attached to the ends A and C and then passes over a small smooth peg. If the angle at the peg is 90° find the inclination of the rod to the vertical.

68. A cone of weight W is held with a slant line vertical and with its apex on a rough plane and then let go. Find

the moment of the forces about the apex, and an expression for the motion assuming the plane so rough that the apex does not slip.

69. By what mechanical method can you distinguish a gilt lead sphere from a hollow gold sphere of the same weight and the same external diameter?

70. A hemisphere resting with its curved surface in contact with a rough horizontal ground and a rough vertical wall is on the point of slipping. Find the inclination of its plane side to the horizontal, the co-efficients of friction for wall and ground being $\frac{1}{2}$ and $\frac{1}{8}$ respectively.

71. In the first system of pulleys show that when there are four lengths of cord between the blocks, the fixed end of the cord being attached to the upper block, the mechanical advantage is reduced to 2·05 if the ratio of the tensions on the two sides of a pulley is three-quarters and the weight of the movable block is neglected.

72. Calculate the maximum velocity of the bob of a simple pendulum, the length of the suspending wire being 100 cm. and the amplitude 30 degrees.

73. Show that if the mass of the wire be not neglected the value of the time period is

$$2\pi\sqrt{\frac{M\left(\frac{2}{5}r^2 + (l + r)^2\right) + \frac{m}{3}l^2}{\left[M(l + r) + \frac{m}{2}l\right]g}}$$

where

M = mass of bob, m = mass of wire, r = radius of bob (assumed spherical), l = length of wire.

Hence show that usually the following expression will give a sufficient approximation,

$$2\pi\sqrt{\frac{l + r}{g}.\left(1 - \frac{m}{6M}\right)};$$

and that this expression will hold good whatever the shape of the bob, provided that its size is small compared with the length of the wire and r is the depth of its centre of mass below the wire.

74. Four points, S_1, S_3, G, S_2, are in this order on the same straight line through the centre of mass G of a rigid bar. Show that if the times of swing about axes through the three points S_1, S_3, S_2, are nearly equal the value of g can be calculated from the formula

$$\frac{T_1^2 + T_2^2}{2} + \frac{T_1^2 - T_2^2}{2} \cdot \frac{\overline{S_1S_2}}{\overline{S_1S_3}} = \frac{4\pi^2}{g}\overline{S_1S_2}.$$

75. Show that if a blow given at a point A of a rigid body produces a certain velocity at a point B, then the same blow given at B will produce at A the same velocity that was previously produced at B.

76. A circular hoop is suspended by an axis passing at right angles to its plane and through its circumference. Find the time period if it is set swinging with a small oscillation about the axis.

77. Show by actually plotting against the time two simple harmonic motions of the same period on the same diagram, and adding their ordinates, that their resultant is another simple harmonic motion of the same period.

78. Find in the same way the resultant when taking place in the same straight line of the two simple vibrations $x_1 = 10 \sin \pi t$ and $x_2 = 5 \sin 2\pi t$.

79. Show by plotting $\cos \theta + \mu \sin \theta$ against θ for a given value of μ (say $\frac{1}{3}$) that this expression takes a maximum value for the value of θ whose tangent is μ. Make use of this fact to show that the best angle of traction is equal to the limiting angle of friction.

80. Prove that if a particle move with a velocity of constant *magnitude*, v, along the sides of a regular polygon inscribed in a circle of radius r, then it is not acted upon by any force except at the corners, at each of which it receives an impulse directed towards the centre; and that the sum of the impulses in one second is equal to the square of the velocity divided by the product of the mass of particle into the radius of the circle.

81. A smooth fine wire, in the form of a circle of radius

r, is placed in a vertical position, and a particle in the form of a small ring slides on it under the action of gravity. If the ring has a velocity of u when at the highest point, what is its velocity when it reaches the bottom?

82. The internal diameter of the cylinder of a steam engine is 1 ft. 6 in.; the length of the stroke is 4 ft.; it makes 18 strokes per minute; under what average pressure of steam (per square inch) must it work in order that work at the rate of 75 horse power may be done on the piston?

83. A certain grocer wishing to compensate for the inequality of the arms of his balance weighs half the tea supplied to each customer in the one pan, and the other half in the second pan. Show that the compensation is incomplete.

84. A hoop rolls down an inclined plane of 30 degrees angle; find its linear acceleration.

85. What fraction of its total energy will be rotational at any moment?

86. A sphere, whose centre of gravity is at a distance from the centre of figure equal to $1/n$th of the radius, is placed in stable equilibrium on a rough board, which is then gently tilted. Explain how the sphere will alter its position relative to the board, and how and when equilibrium ceases to be possible.

87. Explain roughly the kind of experiment you would make in order to compare the masses of two bodies, if there were no such thing as gravity.

How can masses be compared by the balance? State the evidence for the assumption on which this method rests.

88. Find the centre of gravity of a right circular cone.

If the cone be placed with its base in contact with a smooth inclined plane, and be prevented from slipping by a horizontal force applied at the apex, give a diagram showing the arrangements of the forces acting on it.

Also, if the inclination of the plane be 45°, and the radius of the base be equal to $\frac{3}{8}$ of the altitude, prove that the cone will be at the point of toppling over.

89. State and prove the relation between work and kinetic energy, in the case of rectilinear motion under constant forces.

A bullet fired with the velocity of 1000 ft. per sec. penetrates a block of wood to the depth of 12 in. Prove that if it were fired through a board of the same wood, 2 in. thick, its velocity on emergence would be about 913 ft. per sec. [Assume the resistance of the wood to the bullet to be constant.]

90. Find the velocities of two given elastic spheres after direct impact.

A ball is droppod from the top of a tower, height h, and at the same time another ball of equal weight is projected upwards with the velocity $\sqrt{2gh}$ from the base of the tower and collides directly with the falling ball.

If the coefficient of restitution be e, prove that the falling ball will, in the rebound, rise to a height short of the top of the tower by $\frac{h}{4}(1 - e^2)$.

91. A train, weighing, with the locomotive, 50 tons, is kept moving at the uniform rate of 30 miles an hour on the level, the resistance of air, friction, etc., being 40 lb. per ton. Find the horse power of the engine.

By how much must this horse-power be increased if the rate is maintained and water is taken up from a trough between the rails to the amount of m lb. per ft. passed over, the height to which the water is raised above the the trough being h ft., and the kinetic energy imparted to the water in the trough, as well as that of the motion of the water taken up, relatively to the tank, being neglected.

92. Two wires are fixed at right angles to one another, and are each inclined at an angle 45° to the horizon, the wires meeting at their highest point. A light string has small smooth rings, each of mass m, fastened to its ends, and these rings slide one on each wire; at the middle

point of the string a heavy bead, of mass M, is fastened, and rests in equilibrium. Prove that the inclination θ of each half of the string to the vertical is given by

$$\tan\theta = 1 + \frac{2m}{M},$$

and find the pressure on each wire.

93. Prove that the centre of gravity of a uniform triangular plate ABC is the same as that of a frame composed of three uniform bars AB, BC, CA, of equal weights.

Also find what loads must be placed at the corners of a triangular plate whose sides are 3, 4, and 5 ft., in order that it may balance on the centre of the inscribed circle, the sum of the loads being 20 lb., and the weight of the plate being neglected in comparison with them.

94. Determine the mechanical advantage of a pair of nut crackers.

A lever can turn about a fixed point at one end, and rests without friction against a fulcrum $\frac{1}{n}$ of its length from that end, so as to be inclined at an angle a to the horizontal; show that when a load P is suspended from the free end, which is so great that the weight of the lever itself is negligible, the lever will press against the pin of the joint with a force equal to

$$P\,\{(n-1)^2\cos^2 a + \sin^2 a\}^{\frac{1}{2}};$$

and give a construction for the direction of this force.

95. A pulley-block is suspended from a peg by a loop of cord passing round it, of such length that the straight portions next the peg make angles a with the vertical. The block is divided along a vertical diameter into two halves which rest against each other. Show that when the weight of the block and the load suspended from it is W, the two halves of the block are pressed together with a force

$$\tfrac{1}{2}\,W\,(\sec a + \tan a).$$

96. Explain how atmospheric pressure acts upon a plane area immersed under water. A triangular lamina, having an area of 4 sq. ft., has its vertices immersed at depths of 1, 2, and 3 ft. respectively in water. Find the resultant pressure on the area, the atmospheric pressure being taken at 14·7 lb. weight per square inch.

97. Define the metacentre of a floating body such as a ship, and show how the stability depends on its position.

98. In a ship of total displacement M tons and metacentric height h ft., a gun of mass m tons is moved a distance l ft. across the deck. Prove that this will cause the ship to heel over through a small angle whose circular measure is approximately ml/Mh.

99. A sculling boat is propelled by two sculls, each 9 ft. long, and 6 ft. from the rowlock to the blade; and the sculler pulls each hand with a force of 20 lb. Prove that the thrust on each rowlock is 30 lb., but the propulsive force on the boat is 20 lb., and that the boat moves about double as fast as the hands pull.

Estimate the horse-power exerted at 8 miles an hour.

100. Determine the condition of equilibrium of a body movable about an axle.

Prove that the pull exerted by a man in a tug of war is $\frac{a}{b}$ of his weight, where a is the horizontal overhang of his centre of gravity backwards over his heels and b is the height of the rope above the ground.

101. Determine the thrust of water of given depth on one side of a lock gate of given breadth.

Prove that if a lock is closed by two equal gates meeting at an angle α, the thrust between the gates is the same as the thrust on each hinge post, and is $\frac{1}{2}$ cosec α of the hydrostatic thrust which would be exerted on a single gate closing the lock at right angles to its length.

102. Show graphically that a weight of W tons suspended from a point C of a crane ABC against a wall AB, in which the bar BC is horizontal, will set up a thrust of

$\frac{BC}{AB}W$ tons-weight in BC, and a pull of $\frac{AC}{AB}W$ tons-weight in AC.

If the weight W is being hoisted by a chain passing up through a pulley at C, and coming down again at an angle β to the horizon, prove that the additional thrust in BC and pull in AC are

$$\frac{\sin(\alpha+\beta)}{\sin\alpha}W \text{ and } \frac{\sin\beta}{\sin\alpha}W,$$

α denoting the angle ACB.

UNIVERSITY OF LONDON

SPECIAL INTERMEDIATE EXAMINATION AT UNIVERSITY COLLEGE, 1904

Morning

1. To a person travelling at 8 miles an hour, along a road tending west, the wind appeared to come from the north-west. When he stood still it seemed to shift 5° to the north. Find the velocity of the wind. [$\sin 5^\circ = {\cdot}0872$.]

2. A stone is thrown into the air at an angle of 45° to the horizontal and with a speed of 50 ft. per sec. Find the magnitude of its displacement at the instant at which its velocity is horizontal.

3. Find the force which must be exerted, by a man in a lift, on a body of 2 lb. mass which he holds in his hand, to prevent it moving relatively to the lift when the latter is forced to move (*a*) with a uniform speed of 2 ft. per sec., (*b*) with an upward acceleration of 3 ft. per sec. per sec., (*c*) with a downward acceleration of 33 ft. per sec. per sec.

4. Find the horizontal force which must be exerted on an engine of 20 tons while it is going round a curve of 600 yd. radius at the rate of 30 miles per hour. Give the direction as well as the magnitude of the force.

5. Three forces represented completely by the sides of a triangle taken in order act on a rigid body. Will they keep it in equilibrium? If they were moved so as to act on a particle, would they keep the latter in equilibrium? Explain thoroughly.

6. Find in direction and magnitude the least force which will move a body of m lb. up a plane inclined at an angle of θ°, the coefficient of friction being μ.

7. A string is attached to a weight of 10 lb., which lies on a smooth horizontal table; the string after passing over a pulley at the edge of the table has a weight of 1 lb. attached to it. Show that if the weights be free to move

the acceleration of their centre of mass is constant in magnitude and direction, and find its value.

8. Define simple harmonic motion, and describe its character.

The period of the simple harmonic motion of a point is 20 sec. and its maximum velocity is 10 ft. per sec. Find its velocity at a distance of 20 ft. from the mean position.

9. A uniform circular disc (mass = 5 grams, radius = 10 cm.) is started spinning about an axis through its centre and perpendicular to its plane. What was the average couple producing the rotation, if at the end of 5 sec. from starting it rotates at the rate of 16 revolutions per sec. ?

10. Methode der doppelten Wägung: Der abzuwiegende Körper wird in die eine Wagschale gelegt, und in die andere Sand, bis Gleichgewicht hergestellt worden ist. Hierauf wird der Körper aus der Schale herausgenommen, und statt seiner werden so viel Gewichtsstücke in die Schale gelegt, bis wieder das Gleichgewicht hergestellt worden ist.

Translate this passage, and point out the advantages of the method described.

11. Lorsqu'on soumet un corps solide à des efforts croissants, on observe deux périodes bien distinctes: (1) Tant que les efforts ne dépassent pas certaines valeurs, les déformations leurs sont sensiblement proportionelles et disparaissent avec ces efforts. On appelle ces déformations, *déformations élastiques.* (2) Au delà de certaines valeurs des efforts, les déformations croissent plus rapidement; si l'on vient à supprimer les efforts, elles ne disparaissent qu'en partie. Elles sont alors en partie *élastiques*, en partie *permanentes.*

Translate, and give any illustration of these statements with which you are acquainted.

Afternoon

1. Show that it is easier to lift a body a given height, than to drag it up an inclined plane of that height by a rope parallel to the plane, if the coefficient of friction is greater

than the ratio which the difference between the length and height of the plane bears to its base.

2. Find the greatest speed an engine of 100 horse-power can give a train of 70 tons on an incline of 1 in 100, friction being equivalent to a force of 8 lb. weight per ton.

3. Show that the centre of oscillation and point of suspension of a compound pendulum are interchangeable—that is, that moving the former to the latter point moves the latter to the former point.

4. Calculate the spin-radius of a long thin rod with respect to an axis passing through the centre of mass at right angles to the length of the rod. If the axis is moved parallel to itself through the distance a, by how much will the spin-radius have altered?

5. Define the terms "Young's modulus," "torsional couple"; and show how the quantities denoted thereby can be experimentally determined.

6. A cylinder of radius 1 cm. rolls down an inclined plane of 30° angle. Calculate its acceleration.

7. A pair of compasses with a rough joint rests on a smooth, horizontal, cylindrical peg. Show how to calculate the total action due to the forces at the joint.

8. Show how to calculate in magnitude and position the resultant force on one side of a plane rectangular surface immersed in a fluid, one edge of the plane being in the free surface and the pressure at this free surface being zero.

9. A hemispherical vessel is filled with water and is placed with its flat side vertical. Calculate the magnitude and position of the force on the curved side.

10. Define the term "metacentre," and show how its position can be experimentally determined.

INTERMEDIATE EXAMINATION IN SCIENCE
PASS EXAMINATION

Morning

1. Determine the force which will balance two given parallel forces.

Determine graphically the line of action of the force which will balance two given forces when they are so nearly parallel that their point of intersection is a long way off the diagram.

2. The moment about any axis of three forces in equilibrium is zero; and thence also of any number of balancing forces.

Prove that a horse weighing W lb. can exert a horizontal pull of Wa/h pounds at a height h above the ground by advancing his centre of gravity a distance a in front of the position when standing upright on his legs.

3. Determine the centre of gravity of a thin hemispherical shell, and a solid hemisphere.

Prove that a tumbler of thin glass, composed of a cylindrical part and a hemispherical base, will stand upright on a table if the height does not exceed the diameter, and liquid poured in will not disturb the equilibrium before it reaches the cylindrical part.

4. Draw the stress diagram of a span of a symmetrical bridge, in which the roadway is horizontal and divided into equal bays, supported by light steel ropes, attached to the top of the nearest pier, the load being supposed concentrated uniformly at the points of attachment to the roadway.

Draw the diagram for any intermediate stage of the construction, the bridge being built out from each pier.

5. State and prove the formulæ required in rectilinear motion with constant acceleration and retardation.

If an express train from London is doubled, the first half being given 5 minutes' start, and attaining its maximum booked speed of 48 miles an hour in a mile, prove that the two halves will run about 4 miles apart, but the first half will have gone 3 miles before the start of the second half.

6. Prove that if, in the absence of resistance, the time of flight of a shot is T seconds over a range of X feet, the elevation is

$$\tan^{-1}\frac{1}{2}gT^2/X;$$

and determine the maximum height, and the velocity of projection.

Prove that the claim for a rifle, that the bullet does not rise more than one inch in a range of 100 yards, implies that the velocity must be greater than 2,078 feet per second.

7. How long, in time and distance, will a bicycle run freely down an incline of one in 16 before attaining a velocity of 15 and 30 miles an hour.

Prove that the time occupied on a railway worked by gravity between two stations at the same level, connected by two equal inclines of length a feet, descending a depth h feet, is

$$\frac{4a}{\sqrt{2gh}} \text{ seconds,}$$

the train passing the junction of the inclines without shock, and neglecting passive resistances.

8. Prove that the time of a single swing of a simple pendulum of length l ft. is undistinguishable from $\pi\sqrt{(l/g)}$ sec., when the amplitude of oscillation is small.

Prove that if the inclined planes of the last question are replaced by a single circular arc, the time is reduced to $\pi a/\sqrt{2gh}$ sec.

9. Prove that a liquid at rest under gravity is stratified in horizontal plane surfaces of equal pressure and density; and that the pressure increases uniformly with the depth when the liquid is homogeneous.

Calculate the specific gravity of the air, and its specific volume in cubic ft. per lb. implied in a barometer gradient of 1 in. per 1000 ft. vertical, taking the specific gravity of mercury 13·5.

10. Explain the theory of the graduation of the common hydrometer, and prove that the stability increases with the density of the liquid.

Prove that the bulb must have $10\frac{1}{3}$ times the volume of the stem for reading specific gravity between 1·24 and 1·36.

Afternoon

1. Determine the condition of equilibrium of a lever, and give some familiar illustrations.

An eight-oar boat, with oars 12 ft. 6 in. long and divided by the rowlock at 3 ft. 6 in. from the handle, is propelled at 12 miles an hour by each man pulling with a force of 18 lb.; determine the resistance of the water and the horse-power exerted by each man, supposing the forward swing of the bodies to keep up the way between the strokes.

2. Investigate the mechanical advantage of the smooth screw.

Prove that if a man who can apply a force of 28 lb. is provided with a lever 3 ft. long to tighten up iron screw bolts 1 in. in diameter, he will be apt to break the bolts if the number of threads to the inch exceeds 8, the breaking tension of the iron being taken as 30 tons per square inch.

3. Enunciate the triangle of velocity.

It is observed that the streaks of rain on the glass of a railway carriage change from an inclination of 30° to the horizontal when the train is standing at a station to 15° when the train has reached 30 miles an hour. Prove that the vertical velocity of the raindrops is 22 ft. per sec.

4. Prove that the shortest time from rest to rest in which a chain, which can bear a steady load of P tons, can lift or lower a weight of W tons through a vertical distance of h ft. is

$$\sqrt{\left(\frac{2h}{g}\,\frac{\mathrm{P}}{\mathrm{P}-\mathrm{W}}\right)}\ \text{sec.}$$

Up an incline of 1 in 100, from rest at one station to stop at the next a mile off, an engine which can exert a constant tractive force of 4 tons can take a train weighing 240 tons, without using the breaks, in about $4\frac{3}{4}$ min.

5. Prove that, if a body is thrown into the air, the centre of gravity will, in the absence of resistance, describe a parabola.

Two balls weighing W_1 and W_2 lb. are connected by a thread a ft. long, and W_1 is held in the hand while W_2 is whirled round. Determine the motion which ensues if W_1 is released from rest when W_2 is moving with velocity V at inclination α, and prove that in the air the tension of the thread is

$$\frac{W_1 W_2}{W_1 + W_2} \frac{V^2}{ga} \text{ lb.}$$

6. Determine the velocity and the period of revolution in a conical pendulum, when the weight is swung round at the end of a thread a ft. long in a horizontal circle of radius b ft.

Determine the speed in miles an hour suitable for a circular bicycle track of 80 laps to the mile, banked at a slope of 45°.

7. Prove that an interchange, without loss, of momentum takes place in the collision of bodies; but that there is in general a dissipation of energy.

If a bullet weighing w lb. is fired with velocity v at a body weighing W lb., advancing with velocity V, the body will retain the velocity

$$\frac{WV - wv}{W + w}, \text{ or } V - \frac{w}{W}(v - u),$$

according as the bullet is imbedded, or perforates and retains a velocity u.

Calculate the energy liberated, and thence infer the average resistance of the body from the length perforated by the bullet.

8. A funnel in the shape of a symmetrical double cone with communication at the vertex is fitted watertight on a

horizontal plane, and filled with liquid. Prove that liquid will escape unless the funnel is weighted to double the weight of the liquid.

Explain how the ratio of the upward thrust to the weight of liquid may be increased indefinitely (the Hydrostatic Paradox).

9. Explain the Principle of Archimedes in Hydrostatics, applying it to the conditions of equilibrium of a ship.

A steamer loading 25 tons to the inch in fresh-water dock is found, after a 10 days' voyage, burning 52 tons of coal a day, to have risen 25 in. in sea-water.

Determine the displacement in tons, taking 35 and 36 cubic ft. per ton as the specific volume of salt and fresh water.

10. Define work and horse power, and prove that 1 horsepower is the equivalent of 375 mile-pounds per hour.

Calculate the horse-power required to drive a motor car weighing 1 ton up an incline of 1 in 14 at 24 miles an hour, supposing it to reach the same velocity when running freely down this incline.

INTERMEDIATE SCIENCE, 1903

Morning 10 *to* 1

1. Show how to determine the resultant of two parallel forces by a graphical construction.

Extend the method to finding the resultant of three or four parallel forces.

2. Determine the thrust between the buffers of two railway carriages when the screw coupling is turned by a force of P lb. at an arm of a ft., the right- and left-handed screw of the coupling each having n threads to the inch.

3. Prove that a chain made of equal straight links will hang in equilibrium in a vertical plane with the links parallel to lines drawn from one point to equidistant points on a vertical line; and determine graphically the stress at a joint.

4. Calculate the time in seconds at which a body will be at a given height y ft., when projected vertically with given velocity V f/s. Determine the greatest height ascended, and the whole time in the air.

A body was observed to take t sec. in falling past a hatchway to the bottom of a hold h ft. deep. Prove that it fell

$$\frac{1}{2}g\left(\frac{h}{gt}+\frac{1}{2}t\right)^2 \text{ ft.,}$$

and struck with velocity

$$\frac{h}{t}+\frac{1}{2}gt\text{ , f/s.}$$

5. Prove that the centre of gravity of a body thrown in the air describes a parabola, in the absence of resistance.

If a circular hoop is projected, spinning steadily without wobbling, the centre describes a parabola, and the tension of the rim is the weight of a length v^2/g of the rim, where v denotes the rim velocity relatively to the centre of the hoop.

6. Write down the formula for the period of a simple pendulum, and calculate the % of change due to one % change in length or gravity or both.

Calculate the length of a pendulum to beat time to 100 paces a minute; and the speed of march in miles an hour, with a pace of 30 in.

7. Determine the period and velocity of a conical pendulum, in which the plummet is swung round by a thread l ft. long at an angle α to the vertical.

Calculate the length of the equivalent simple pendulum making small plane oscillations.

8. Define the specific gravity (S.G.) of a substance; and calculate the S.G. when V litres weigh W grammes.

Calculate in metres the pressure head or height of the homogeneous atmosphere, when the density of the air is 1·23 grammes per litre at a pressure of 1 kg/cm.^{2}.

9. Discuss Archimedes' principle concerning the buoyancy of a fluid, and give illustrations.

Prove that the buoyancy of a life-buoy made of cork of S.G. s weighing W lb. is

$$W\left(\frac{1}{s} - 1\right) \text{lb.}$$

10. Describe the theory of the common hydrometer for determining the S.G. of a liquid.

What is the reading of the hydrometer when the liquid contains

(i) a substance in solution,
(ii) a substance in suspension as a precipitate,
(iii) when the liquid is aërated and effervescent, or boiling with small rising bubbles?

Afternoon, 2 *to* 5

1. Prove that the moment about any axis of three forces in equlibrium is zero.

Prove that if a straight bar supported at the ends is strained by a force, the moment at any point tending to bend the bar is proportional to the distance from one end or the other of the bar.

2. Draw the stress diagram of a triangular crane, ABC, of which AB is a vertical post, and AC a horizontal beam supported by a tie-rod BC, due to a load W tons carried at any point D of BC.

Prove that the bending moment at D is

$$W \frac{AD \cdot DC}{AC} \text{ foot-tons.}$$

3. Determine the change of the centre of gravity of a body weighing W lb. when a part weighing P lb. is removed from one position to another.

Discuss the tendency of a spherical body, or some liquid, to spill out of a scale pan, suspended symmetrically from a point by three equal chains, the scale pan being curved to a spherical surface.

4. Investigate the formulæ

$$\text{(i) } at = v - V, \text{ (ii) } as = \tfrac{1}{2}v^2 - \tfrac{1}{2}V^2,$$

for rectilinear motion with constant acceleration a.

Prove that if a motor car going at 100 kilometres an hour can be stopped in 200 metres, the breaks can hold the car on an incline of about 1 in 5; and determine the time required to stop.

5. Prove the dynamical formulæ

$$\text{(i) } Pt = \frac{Wv}{g} - \frac{WV}{g}, \text{ (ii) } Ps = \frac{Wv^2}{2g} - \frac{WV^2}{2g},$$

explaining the notation.

Calculate the time and distance in which the velocity of a 100 lb. shot will fall from 2,010 to 1,990 f/s, against an average air resistance of 800 lb., neglecting gravity.

6. Prove that if the graduations of a spring balance are uniform, a weight suspended will perform vertical oscillations which are simple harmonic.

Determine the length of the equivalent simple pendulum.

7. Prove that if a hammer weighing W lb., striking a nail weighing w lb. with velocity V f/s, drives it a ft. into a fixed block of wood, the average resistance of the wood in pounds to the penetration of the nail is

$$\frac{W^2}{W + w}\frac{V^2}{2ga}.$$

If, however, the block is free to recoil and weighs M lb., the resistance obtained would be

$$\frac{MW^2}{(M + W + w)(W + w)}\frac{V^2}{2ga}.$$

8. Prove that the increase of pressure for every foot vertically downwards in liquid of S.G. s is

$$0{\cdot}433 \times s \text{ lb./in.}^2.$$

Mining in ground of uniform density at a depth of h ft. determine the % of coal that can be won, leaving sufficient as pillars for the support of the roof, supposing the coal to crush under its own weight in a column k ft. high.

9. Describe the barometer; and calculate the S.G. of mercury when the barometer stands at 75 cm. for an atmospheric pressure of 1 kg/cm.^{2}.

Prove that the weight of the atmosphere is practically the same as of an ocean of mercury covering the earth with average depth of the barometric height; and prove that this weight is about

$$5 \times 10^{18} \text{kg}.$$

10. Prove that the horse-power of an engine which overcomes a resistance of R lb. at a speed of S miles an hour is

$$RS \div 375.$$

Calculate the power required to drive a motor car weighing W tons at S miles an hour,

(i) on the level,

(ii) up an incline of 1 in m,

when the car is found to get up to the same speed in running freely down this incline.

INTERMEDIATE (EXTERNAL), 1904

Morning

1. Explain a graphical method of finding the resultant of a number of given forces acting on a particle.

A light string of length l has its ends fixed at two points A, B, at a horizontal distance a apart, and a heavy ring of weight W can slide on the string. Prove that the ring can rest vertically beneath B if a force Wa/l be applied to it parallel to AB.

2. A man stands on a ladder which leans against a vertical wall. Assuming the pressure on the wall to be horizontal, find (geometrically) the horizontal thrust of the

foot of the ladder on the ground, having given the total weight of the man and ladder, and the line of action of this force.

For example, let the length of the ladder be 15 ft., the distance of its foot from the wall 5 ft., and let the total weight be 3 cwt., acting in a line which cuts the ladder one-third of its length from the foot.

3. Apply the principle of work to find the pressure which can be exerted by a smooth screw in the direction of its axis, when a given twisting couple acts on it.

For example, suppose the screw to have 4 turns to the inch, and the couple to be 18 ft.-lb.

4. A triangular frame of jointed rods ABC, right-angled at A, can turn freely about A in a vertical plane. The side AB is horizontal, and the corner C rests against a smooth vertical stop below A. Find (graphically or otherwise) the stresses in the various bars due to a weight W suspended from B.

Determine the stresses numerically when $AB = 3$ ft., $AC = 1$ ft., $W = 50$ lb.

5. A pile is being driven in by repeated impacts of a falling weight. How does the extent to which the pile is driven in by each blow depend (1) on the magnitude of the weight, and (2) on the height to which it is raised before being released? Give reasons for your answers.

If the weight be 1 ton, and the height from which it falls be 10 ft., and the pile be driven in a tenth of an inch, find the resistance in tons.

6. Find the acceleration of a particle down a smooth inclined plane, stating clearly the principles assumed in the proof.

A railway carriage runs with a uniform velocity of 10 miles an hour down an incline of 1 in 250, and on reaching the foot of the incline runs on the level. Find how many yards it will run before coming to rest, assuming the resistance to be constant and the same in each stage of the motion.

7. Find the pressure on the pulley in the simplest ideal

form of Atwood's machine, and prove that it is never greater than the sum of the suspended weights.

A uniform chain of length l and weight W passes over a smooth peg, hanging vertically on each side. If the chain be running freely, prove that when the length on one side is x, the pressure on the peg is

$$\frac{4x(l-x)}{l^2}W.$$

8. Prove that the small oscillations of a pendulum are simple-harmonic, and find the period in terms of the length.

If the pendulum swing from rest to rest through a space of 8 in. in 3 sec., find its length. Also (in feet per sec.) its velocity at the lowest point.

9. Give and prove a rule for finding the total pressure of a liquid on one side of a plane area.

Two liquids which do not mix are contained in a vessel with vertical sides. Their specific weights are w_1, w_2, and their depths are h_1, h_2 respectively; find the mean pressure-intensity on the sides of the vessel.

Represent by a graph the variation of pressure-intensity with the depth.

10. A solid lighter than water is tied to the bottom of a cistern by a string. Prove that the string will be vertical, and give a diagram of the forces acting on the body. [The centres of gravity and of buoyancy, and the point of attachment of the string, are not to be assumed to be in the same vertical.]

Afternoon

1. Explain how to find by resolution and moments the magnitude and line of action of the resultant of any number of given coplanar forces acting on a rigid body.

Apply this method to the following case:—

ABC is a triangle having a right angle at B. The side AB = 12 in., BC = 5. A force of 52 lb. weight acts from A to C, one of 24 acts from B to A, and one of 27 from

C to B; find the magnitude and line of action of the resultant.

[Draw the figure to scale and exhibit the resultant.]

2. Define the *coefficient of friction* and the *angle of friction* for two rough bodies.

A mass of 500 lb. on a rough inclined plane, for which the coefficient of friction is $\frac{1}{5}$ and whose inclination is $\tan^{-1}\frac{1}{2}$, is sustained by a rope which is pulled in a horizontal direction; prove that the greatest and least tensions of this rope consistent with equilibrium are about 389 and 136·4 lb. weight respectively.

3. State the necessary and sufficient conditions of equilibrium of a system of coplanar forces acting on a rigid body.

A uniform bar, AB, 10 ft. long, of weight W, is freely movable in a vertical plane about a smooth axis fixed at A; it is sustained at an inclination $\tan^{-1}\frac{3}{4}$ to the horizon by resting against a fixed peg at C, where AC = 6 ft. Find the magnitudes and exhibit the lines of action of the pressures at A and C.

In what way is the problem modified if the axis at A is rough?

4. Define the *centre of mass* (or gravity) of a system of particles, and explain any method of finding it.

Above a horizontal plane there are particles whose masses are 3, 5, 2 grm. at distances of 6, 2, 4 in. respectively from the plane; below the plane are particles 4 and 2 at distances 3 and 4 respectively from the plane. What is the distance of their centre of mass from the plane?

5. Enunciate the general principle of buoyancy, and state the conditions that are satisfied when a body floats under the action of gravity alone in a liquid (neglecting considerations of stability). If the specific gravity of ice is ·918, and that of sea-water 1·026, show that of the whole volume of an iceberg $\frac{2}{19}$ will be above water.

6. A solid homogeneous cone of height 6 ft. and diameter of base 4 ft. is allowed to float with its axis vertical and vertex downwards in a cylindrical tank, the diameter of

whose cross-section is 6 ft., the tank containing a liquid whose specific weight is to that of the cone in the ratio 8 : 3; how much does the level of the liquid rise, and how much of the axis is immersed?

7. Deduce (preferably by means of a diagram) the relation between the *distance* travelled and the *time* in the case of uniformly accelerated rectilinear motion.

A train is moving through a station with a speed of 6 miles per hour, while its motion is being accelerated at the rate of 3 miles per hour per minute. If this rate continues, how far will the train be from the station after 20 minutes?

8. Define the *impulse* and the *work* of a constant force.

A body, whose mass is 5 lb., moving with a speed of 160 ft. per sec., suddenly encounters a constant resistance equal to the weight of $\frac{1}{4}$ lb., which lasts until the speed is reduced to 96 ft. per sec.; for what time and through what distance has the resistance acted?

9. A uniform chain 6 ft. long, having a mass of 2 lb. per ft., is laid in a straight line along a rough horizontal table for which the coefficient of friction is $\frac{1}{2}$, a portion hanging over the edge of the table so that slipping is just about to occur. If a slight disturbance sets the chain slipping, find the tension at the edge of the table when x ft. have slipped off.

10. Prove that the path of a particle which is projected in any way at the surface of the earth, and acted upon by no force but its own weight, is a parabola.

A, B, C are three points in order in a horizontal line, AB being 640 ft.; a particle is projected from A with a velocity of 390 ft. per sec. in a direction making $\tan^{-1}\frac{5}{12}$ with AC; at the same instant another particle is projected from B with a velocity of 250 ft. per sec. in a direction making $\tan^{-1}\frac{3}{4}$ with BC; show that these particles will collide, and find when and where.

INTERMEDIATE (INTERNAL), 1904

Morning

1. Two forces acting at a point O are represented in magnitude and direction by k . OA and l . OB. Show that they are equivalent to a force $(k + l)$. acting OC along OC, where C is a point in AB, such that

$$l \,.\, \mathrm{BC} = k \,.\, \mathrm{AC}.$$

If the directions of the two forces are parallel, state the corresponding theorem.

2. Under what circumstances is the algebraic sum of the moments of a system of co-planar forces zero with respect to a given point? What would you know about a system of forces in a plane if

(1) the sum of the moments of the forces with respect to two points A and B were zero;

(2) the sum of the moments of the forces with respect to three points, A, B and C (not in the same straight line) were zero.

A triangle is inscribed in a circle, and forces proportional to the sides act along the sides in cyclical order. Find the magnitude of two forces touching the circle which will be in equilibrium with the given forces.

3. State the necessary and sufficient conditions for the equilibrium of three forces (1) acting at a point (2) acting on a body.

Forces proportional to the sides of a triangle act

(1) Through the centre of the inscribed circle perpendicularly to the corresponding sides;

(2) Through the corresponding vertices parallel to the sides taken in order.

Find in each case whether the forces are in equilibrium, and if not the resultant.

4. Find the centre of mass of the figure formed by cutting a square of side a out of a circular area of radius b,

the point of intersection of the diagonals of the square being at a distance l from the centre of the circle.

Find the total fluid pressure on such an area when placed in a liquid of density ρ, the centre of the circular area being at a depth k, and the line joining this point to the intersection of the diagonals being inclined at an angle α to the surface of the liquid.

5. State the laws of friction.

On the side of a hill whose face is inclined at an angle of 60° to the vertical, a man supports himself with the aid of a stick which is held so as to be always perpendicular to the face of the hill. If the distance of his centre of gravity from his feet is 3 ft., and if the distance of the point of his body which rests on the end of his stick is 4 ft., and the coefficient of friction between his feet and the face of the hill is $\frac{2}{3}$, show that he can, without slipping, move his body forward from the vertical through an angle $\tan^{-1}\frac{2-\sqrt{3}}{3}$ towards the upward direction of the hill.

6. The vertical height of the inclined plane in a water chute is h feet. If the mean retardation of the water to a boat of mass M is λ, find the distance through which the boat will go before it comes to rest and the time from starting. Assume that the chute is curved at the bottom so that velocity down the chute is converted into horizontal velocity.

7. A cricket ball is bowled with a velocity v and reaches the player's bat moving in a direction making an angle α with the ground. If the impulse given to the ball by the bat is horizontal, find its least magnitude so that the ball may be driven back a horizontal distance l feet before it touches the ground.

8. A small body of mass m is rotated in a vertical plane at the end of a string of length l, with an angular velocity ω.

Find (1) the height to which the body will rise if the string be released when it is in a horizontal position; (2) the distance in a horizontal direction and the height to which it will go if released when the string is inclined at an angle $\frac{3\pi}{4}$ to the downward drawn vertical.

9. A bullet of mass m is fired horizontally at a block of wood of mass M, in a straight line passing through its centre of mass, and becomes imbedded therein. If the velocity of the bullet be v and the block of wood be suspended so that its centre of gravity can swing at a distance l from a point of suspension, find through what angle the block of wood will swing if its mass may be supposed collected at its centre of mass.

Find also the momentum of the recoil of the gun.

10. To the end of a vertical column a horizontal arm l ft. long is attached, and from the end of the arm a car in the shape of a boat is suspended in a horizontal position a ft. below the end of the arm. The car is able to swing only in the plane through the vertical column and the horizontal arm. Find the rate at which the column must rotate if the line joining the centre of the car to the end of the arm becomes inclined at an angle 45° to the vertical (downwards drawn).

11. Unter welchen Gleichgewichtsbedingungen schwimmt ein an einer Schnur gehängter Körper?

A rod 8 ft. long (specific gravity $\frac{7}{8}$) is floating in water supported by a string attached to a point 1 ft. from the end. Find the length of the rod immersed in the water.

12. Find the periodic time in a simple harmonic motion under the action of a force $\mu . \mathrm{OP}$ directed from a variable point P to a centre O.

Deduce the time of a complete vibration of a simple pendulum of length l making small vibrations, and show what conditions must be fulfilled if the length of time of the vibrations is to remain constant.

Afternoon

1. Prove the truth of the Parallelogram of Forces when the two forces act upon a particle.

Two forces are in the ratio of m to n, and their resultant is perpendicular to the second force. Show that the resultant is

$$\sqrt{\frac{m^2}{n^2} - 1} \text{ times the second force.}$$

Find the resultant of two given parallel forces whose directions are the same.

A bookshelf is 3 ft. long, and is supported at its two ends; starting from one end it carries 7 books, each 1½ in. thick and weighing 1 lb. 5 oz., then 9 books each of thickness 2 in. and weight 1 lb. 2 oz., and then 6 books each of thickness 1¼ in. and weight 17 oz. Find the reactions at the points of support.

3. If three forces keep a rigid body in equilibrium, state what is known about the lines of action of the forces.

A uniform rod is hung up by means of two strings tied to its ends, the other extremities of the strings being attached to a fixed point. Prove that the tensions of the strings are proportional to their lengths.

4. In the case of a body supported by means of a smooth screw, find the mechanical advantage.

State the advantages possessed by a Hunter's differential screw, and in this case find also the mechanical advantage.

5. Explain what is meant by the terms *coefficient of friction*, *angle of friction*.

A heavy beam AB rests at a given inclination α to the horizon with one end A in contact with a rough horizontal plane, and is supported by a light cord tied to the other end B. Show that the beam is in limiting equilibrium when the inclination of the cord to the horizon has either of the values

$$\tan^{-1}\left[\frac{1}{\mu} \pm 2 \tan \alpha\right],$$

where μ is the coefficient of friction.

Show that the corresponding normal reaction of the plane is

$$\frac{W}{2} \cdot \frac{1}{1 \pm \mu \tan \alpha}.$$

6. Find the space s described in time t by a body which starts with velocity u and moves with constant acceleration f in the direction of its original motion.

A motor-car travelling at the rate of 20 miles per hour is stopped by its brakes in 4 sec.; show that it will go about 20 yards from the point at which the brakes are first applied.

7. Two heavy unequal particles are connected by a light string which passes over a small smooth pulley. Find the resulting acceleration.

Two particles, of masses M and m, are connected by a string, of length $2a$, which passes over a smooth pulley. Initially the particles are held close to the pulley, one on each side of it, and are let drop at the same instant. When the string tightens, show that the kinetic energy destroyed is

$$\frac{4Mm}{M+m}ga,$$

and that the impulse in the string is

$$\frac{2Mm}{M+m}\sqrt{2ga}.$$

8. Explain what is meant by the Principle of the Physical Independence of Forces, and point out its application to the motion of projectiles.

A boy throws a stone so as to hit an object which is at a horizontal distance a and at a vertical height h. If V be the velocity of projection, show that the inclination α of the direction of projection is given by the quadratic equation

$$\tan^2\alpha - \frac{2V^2}{ag}\tan\alpha + 1 + \frac{2V^2h}{a^2g} = 0.$$

9. Une particule attachée par une ficelle à un point fixe fait la révolution d'un pendule conique ; trouver la tension de la ficelle et prouver que le temps de la revolution est pareil à celui d'un pendule simple dont la longueur égalerait la distance verticale que sépare la particule du point fixe.

10. A particle is projected along the inner surface of a smooth vertical surface of radius a, its velocity at the lowest point being $\frac{1}{5}\sqrt{95ga}$. Show that it will leave the circle at an angular distance $\cos^{-1}\frac{3}{5}$ from the top and that its velocity then is $\frac{1}{5}\sqrt{15\,ga}$.

11. Define carefully what is meant by the *density* and *specific gravity* of a substance.

Common lead pipes, of length 15 ft., of bore $\frac{1}{2}$ in. in diameter, and of thickness ·2 in., weigh 16 lb. each. Find the specific gravity of the material $\left[\pi = \frac{22}{7}\right]$.

12. Establish the principle of Archimedes.

If W_1 be the weight of a body in water, and W_2 its weight in air of S.G. s, show that its true weight in *vacuo* is

$$\frac{W_2 - sW_1}{1 - s}.$$

UNIVERSITY COLLEGE

ENGINEERING MATRICULATION EXAMINATION, 1904

1. A 1200 lb. shot, moving with a velocity of 2000 ft per sec., impinges on a plate of 10 tons, passes through it, and goes on with a velocity of 400 ft. per sec. If the plate be free to move, find the velocity it acquires.

2. A train travels from rest at one station to rest at another 5 miles distant in 8 min. Find the maximum velocity, assuming that the velocity increases uniformly and then decreases uniformly to zero.

3. Prove the triangle law for adding velocities. If the velocities are 2 and 3 ft. per sec. respectively, making an angle of 30° one with another, find the resultant.

4. Which will travel farthest from rest in a given time, a mass of 100 grams acted on by a force of 100 dynes or a kilogram acted on by a force of 1 gram weight?

5. Find the horizontal force which will keep a 10 gram mass at rest on a smooth inclined plane whose angle is $30.^\circ$

6. What is meant by the term "energy"? Why do we say that a moving body possesses energy?

7. If a triangular lamina ABC weigh 6 oz., what weight must be placed at A in order that the centre of gravity of the whole may bisect the line joining A to the middle point of BC?

TABLES

In the International System of units the prefixes
deci-, centi-, milli-
denote one-tenth hundredth thousandth
part respectively of the unit to which they are attached; thus 1 cm. is the $\frac{1}{100}$ part of a metre.

The prefixes deka-, hecto-, kilo-, mega,-
denote ten, hundred, a thousand, a million
times the unit to which they are prefixed; thus a kilometre = a thousand metres.

SPACE

1 inch = 2·5400 centimetres
1 foot = 30·4801 „
1 mile = 160,935 „
1 square foot = 929·04 square centimetres
1 cubic foot = 28,317 cubic „

MASS

1 lb. avoirdupois = 453·59 grams
15·432 grains = 1 gram

TIME

86400 mean solar seconds = 1 mean solar day
86164 „ „ „ = 1 sidereal day

VELOCITY

1 foot per second = 30·4801 cm. per second
1 mile per hour = 44·704 „ „
1 knot = 51·453 „ „

STRESS

1 lb. weight per sq. foot = ·48824 grm. weight per sq. cm.
= 479 dynes per sq. cm.
1 lb. weight per sq. inch = 70·31 grm. weight per sq. cm.
76 cm. pressure of mercury at 0°C. = 1033·3 grm. weight per sq. cm.
= 1·0136 × 10^6 dynes per sq. cm.

WORK AND ENERGY

1 foot-lb. = 1·3560 × 10^7 ergs
1 joule = 10^7 ergs
1 kilogram-metre = 10^5 grm.-cm. = 9·81 × 10^7 ergs.
1 Board of Trade unit = 1 Kilowatt-hour = 3·6 × 10^6 joules.

POWER

1 horse-power = 7·46 × 10^9 ergs per second
1 force-de-cheval = 7·36 × 10^9 ,, ,, ,,
1 kilowatt = 10^{10} ,, ,, ,,

SPECIFIC GRAVITIES (from Everett, "C.G.S. Units")

METALS

Aluminium	2·6 — 2·8
Brass	8·44— 8·70
Copper	8·80— 8·95
Gold	19·26—19·34
Iron (Grey Cast)	7·03— 7·13
,, (White Cast)	7·58— 7·73
,, (Wrought)	7·80— 7·90
Steel	7·8 — 7.9
Lead	11·34—11·36
Mercury at 0° C.	13·596
Platinum	21·20—21·70
Silver	10·40—10·57
Zinc	7·04— 7·19
Water	1

WOODS

Ash	·65— ·85	Pitch Pine	·83—·85
Beech	·70— ·90	Spruce	·48—·70
Cedar	·49— ·57	Mahogany (Span.)	·85
Cork	·26	Oak	·60—·90
Ebony	1·11—1·33	Willow	·40—·60

LIQUIDS

Alcohol	·791	Oil (Olive)	·918
Ether	·736	Sea-water	1·025
Oil (Linseed)	·94	Sulphuric Acid	1·85

The above are also the **density** in grams per cubic centimetre. To obtain densities in lb. per cubic foot multiply by 62·4.

DENSITY OF GASES

in grams per cubic centimetre at 0°. C, and a pressure of 10^6 dynes per square centimetre.

Air (Dry)	·0012759
Oxygen	·0014107
Nitrogen	·0012393
Hydrogen	·00008837
Carbonic Acid	·0019509

The density of hydrogen being given, that of any other gas is found approximately by multiplying by half its molecular weight.

MISCELLANEOUS DATA

Ratio of circumference to diameter of circle, $\pi = 3·14159265$

$\pi^2 = 9·8696$

Base of natural logarithms, $e = 2·718281828$5

$\log_{10} e = ·4342945$

$\log_e 10 = 2·3025851$

Area of circle of radius $r = \pi r^2$

Surface of sphere $= 4\pi r^2$

Volume of sphere $= \frac{4}{3}\pi r^3$

Volume of cone $= \frac{1}{3} \times$ area of base $\times$ altitude.

TRIGONOMETRICAL RATIOS.

ANGLES.		Sines.	Cosines.	Tangents.
Degrees.	Radians.			
0	·0000	·0000	1	·0000
1	·0175	·0175	·9999	·0175
5	·0873	·0872	·9962	·0875
10	·1745	·1736	·9848	·1763
15	·2618	·2588	·9659	·2679
20	·3491	·3420	·9397	·3640
25	·4363	·4226	·9063	·4663
30	·5236	·5000	·8660	·5774
35	·6109	·5736	·8192	·7002
40	·6981	·6428	·7660	·8391
45	·7854	·7071	·7071	1·0000
50	·8727	·7660	·6428	1·1918
55	·9599	·8192	·5736	1·4281
60	1·0472	·8660	·5000	1·7321
65	1·1345	·9063	·4226	2·1445
70	1·2217	·9397	·3420	2·7475
75	1·3090	·9659	·2588	3·7321
80	1·3963	·9848	·1736	5·6713
85	1·4835	·9962	·0872	11·430
90	1·5708	1·0000	·0000	Infinity

ANSWERS TO EXAMPLES

CHAPTER II. RECTILINEAR KINEMATICS (pp. 30—32)

(1) ·117. (2) 2, $1\frac{1}{2}$, $1\frac{2}{3}$, $1\frac{2}{3}$ feet/min.
(3) 6000 feet/sec. (4) — 5996 cms./sec.
(5) $4·32 \times 10^4$ cm. (6) 100 feet/sec.
(7) 3·26 mile/day, 6·096 cm./sec. (8) 980·85.
(9) 5280 feet/min., 88 feet/sec.
(10) (*α*) 256, 256, — 1536 feet resp.; (*β*) — 92·5 miles
(11) 96. (12) 10·25.
(14) After 12 min., 3120 feet from first man.
(15) $1\frac{1}{9}$ mile/min. (16) 2112 feet.
(17) 10·09 sec. (18) 4 sec., 100 feet/sec.
(19) 484 feet from starting point.
(20) $t = (v_1 - v_2)/g$.

CHAPTER III. RECTILINEAR DYNAMICS (pp. 58—60)

(1) $3\frac{4}{7}$ feet/min.^{2} (2) 1 : 64·4. (3) $4·452 \times 10^5$.
(4) 4 dynes. (5) i. ·71 oz. wt.; ii. 5·71 oz. wt.
(7) i. 56 lb.; ii. 51·2 lb.; iii. 61·8 lb.; iv. ∞.
(8) 2·91 feet/sec.^{2}, 5·45 grm. wt.
(10) Pressure equals tension in string. (11) 23.
(12) Veloc. of 5 grm. is 59·8 feet/sec. down, veloc. of 6 grm. is 68·2 feet/sec. down.

CHAPTER IV. MOMENTUM (pp. 68—70)

(1) Equal. (2) 9 : 19. (3) 15 : 1.
(4) 1010 feet/sec. (5) No.

(6) $3\frac{4}{7}$ in direction of first body.

(7) $\frac{5}{7}$ and $9\frac{2}{7}$, both in opposite directions to original motions.

(8) 2·121 sec., $13\frac{1}{3}$ feet. (9) ·5.

Chapter V. Addition of Vectors (83—84)

(1) i. 9·673 making with 4 angle of 18° 4′
ii. 3·230 „ „ „ 68° 16′
iii. 7·431 „ 5 „ 16° 35′

(2) 64·84 feet/sec. (or 44·21 miles/hour), making with 44 angle of 73° 41′.

(5) 75° 30′. (6) 1·774 miles S. by 7° 52′ W.

(8) 42·50 miles/hour at angle 86° 40′ with A's velocity, 7·391 miles/hour^2 at angle of 22° 30′ with A's acceleration.

(10) 1·077 cm./sec.^{2}, 68° 12′ with horizontal.

(11) ·832 miles (1464 yd.), A 542 yd. east of initial position, B 812 yd. south, 4 min. 37 sec.

Chapter VI. Equilibrium of a Particle (pp. 93—94)

(1) 56,640 dynes. (2) 115·5 grm. (4) 4° 51′.
(6) 1·809 lb. wt. (8) 70·71 lb. (9) 29·29 lb. wt.

Chapter VII. Curvilinear Motion (pp. 109—110)

(1) 219·3 cm./sec.^{2}, 21,930 dynes. (4) About 4·8 in.

(5) 50·2 tons wt. on flange of outer wheel, 30·1 tons wt. on inner.

(8) 75° 37′.

Chapter VIII. Work and Energy (pp. 123—124)

(1) 1000 ft.lb. (2) 500 ft.lb. (3) $3·532 \times 10^{10}$.
(4) 1/330, 1/660, ·00954 H.P. (5) $7·848 \times 10^8$ ergs.
(6) 4000 ft.lb., 0, 2000 ft.lb., 1000 ft.lb.

(7) 0, 4000 ft.lb., 2000 ft.lb., 3000 ft.lb.

(8) Energy of shot = 179·2 energy of gun.

(9) 3656, 3406, 2930 ft.lb.

(10) 68·75 feet/sec. (46·9 miles/hour).

(11) 1.348×10^7. (12) 42650 ft.lb. approx.

CHAPTER IX. CENTRES OF MASS AND THEIR MOTION (pp. 142—144)

(1) 14·4 cm. from end. (3) 4·62, 5·97 in.

(5) $(6 + 5\sqrt{3})/13$ times the side of square measured from apex of triangle along median line.

(6) $\sqrt{3} \times$ side of square.

(7) On join of middle points of parallel sides and $\frac{5}{9}$ its length from shorter side.

(8) 64 feet/sec., 64 feet, 2 sec.

(9) $\left(\frac{m_1 - m_2}{m_1 + m}\right)^2 g$, $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 gt$.

(11) $\frac{m_2\sqrt{m_1^2 + m_2^2}}{(m_1 + m_2)^2} g$, angle with horizontal$=\tan^{-1}(m_2/m_1)$.

(14) 4·8 cm./sec.^{2}

(15) 24, 12, 8, 6 cm. resp. from original positions, centre of mass at distance 14·6 cm. from join of 1 and 4, and 7 cm. from join of 1 and 2. (Forces taken as being parallel to join of 1 and 2.)

CHAPTER X. TORQUE, RESULTANTS, AND COUPLES (pp. 165—166)

(1) Twice the area of the triangles.

(2) Twice area of figure.

(6) 17 units at 35 in. from greater, 7 units in same sense as and 85 in. from greater.

(7) 7·80 from 2 lb. wt.; 6·79 from 3, 5·59 in. from 4.

(8) Between forces 3 and 7 and at distance ·95 cm. from latter.

(9) $30\sqrt{3}$ × force.

(10) If force 12 acts along straight line, to a single force $6\sqrt{3}$ at angle of 30° with straight line, and acting at a point 180 cm. from end where force of 24 units acts.

Chapter XI. Equilibrium of a Rigid Body (pp. 180—181)

(1) Upper joints, $\frac{W}{2}\sqrt{\frac{13}{3}}$, angle with horizontal $= \tan^{-1} 2\sqrt{3}$, lower joint, $W/(2\sqrt{3})$, acting horizontally.

(2) Upper joint, $\frac{W}{2}\sqrt{\frac{31}{3}}$, angle with horizontal $= \tan^{-1}\left(\frac{3\sqrt{3}}{2}\right)$, lower joints, $\frac{W}{2}\sqrt{\frac{7}{3}}$, angle with horizontal $= \tan^{-1}\frac{\sqrt{3}}{2}$.

(3) $\frac{\sqrt{13}}{2\sqrt{7}}W$, angle with vertical $= \tan^{-1}\frac{3\sqrt{3}}{8}$.

(5) $3W/2$ applied at D where $\overline{BD} = \overline{BC}/3$.

(6) $\frac{3W}{2\sqrt{2}}$ and $\frac{W}{2\sqrt{2}}$. (7) $\frac{3\sqrt{3}}{4\sqrt{2}}W$ and $\frac{4+\sqrt{6}}{8}W$.

(8) Angle with horizontal, $\tan^{-1}\frac{a-b}{\sqrt{4c^2-(a+b)^2}}$.

$$F_1 = \frac{Wac}{(a+b)\sqrt{c^2-ab}} \text{ and } F_2 = \frac{b}{a}F_1.$$

(9) 2W. (10) $\frac{3}{2\sqrt{7}}$ × wt. of beam.

Chapter XII. Friction (pp. 197—199)

(1) ½ kilogram wt. (2) 163·5 cm./sec.^{2}

(3) 11·3 lb. wt. (4) nearly 36 grm.

(5) ·447 ton wt. at an angle of 26° 33′ to horizontal.

(6) ·748 ton wt. at an angle of 18° 26′ with plane.

(7) ·728 sec. (10) $50^\circ 12'$.

(11) $86^\circ 50'$ when at bottom, $31^\circ 48'$ when at top.

(12) Whether at top or bottom, the force on wall is 94·8 lb. wt., force on ground is 201·2 lb. wt.

(13) $\tan a_1 = 2/(\cot a_2 - \sqrt{3})$ with condition that neither a_1 nor a_2 may be greater than corresponding limiting angle.

(15) $2·387 \times 10^8$ dynes (243·3 kilograms wt.)

(16) $1·257 \times 10$ ft. lb.

CHAPTER XIII. MOTION ABOUT A FIXED AXLE (pp. 210—212)

(1) 9·204 ft.lb.

(2) ·2 oz. wt. approx.; 1·2 inch-oz. wt. approx.

(4) $1200\frac{2}{3}$ lb.ft.^{2} (5) ·05221. (6) 4000 lb.ft.^{2}

(7) 320 gr.cm.^{2} (8) 2 min. 11·1 sec.

CHAPTER XIV. MOTION OF A RIGID BODY (pp. 221—223)

(1) 14 : 15. (2) $a = \frac{2}{3}\frac{\mathrm{T}}{m} \cdot \frac{r'-r}{r}$

(3) $a = \frac{\mathrm{T}}{m} \cdot \frac{(r-r')r'}{k^2+r'^2}$ in opposite direction to T: (k = spin-radius).

(4) 2·94 cm./sec.^{2} (5) $\theta = \cos^{-1}\frac{r'}{r}$.

(6) 1/144. (7) 2·42 ft./sec.

CHAPTER XV. ENERGY OF A RIGID BODY (243—245)

(1) $1·466 \times 10^7$ ft.lb. (2) 300 ft.lb. (3) 10 ft. tons.

(4) 100 ft.lb. (5) 226·1.

(6) 2112 ft.lb./min. or ·064 H.P. (8) 25/49.

(9) 998 ft.lb. (11) Uniform velocity of $5\frac{1}{3}$ feet/sec.

(12) ·485 ft.lb.

(13) ·125 ft.lb.; ·1212 ft.lb.; 5·82lb. wt. (14) ·970.

(17) ·1005 ft.lb. (18) 3·325 tons wt.

CHAPTER XVI. PRINCIPLE OF VIRTUAL WORK (pp. 254—255)

(See answers to Chapter XI.)

CHAPTER XVII. ELEMENTARY MACHINES (pp. 273—276)

(1) i. 1·396; ii. 1·034; iii. 1·383.
(2) i. ·577; ii. ·268; iii. 1. (3) 20·37 lb. wt.
(4) 13·6 lb. wt. (5) 8. (6) Very nearly 2·6.
(7) i. 14 lb. wt; ii. 16·6 lb. wt.; iii. 27·72 lb. wt.
(10) 71070 grm.cm.^{2} nearly. (11) 6 feet.
(12) 5·18 lb. wt. (13) $\frac{2(Wb - Pa)}{a - b}$. (15) $\frac{2\pi l}{M}$.
(16) 14·3 lb. wt. (17) $\frac{2\mu W}{1 - \mu^2}$.

CHAPTER XVIII. SIMPLE HARMONIC MOTION (pp. 299—300)

(1) 7·78 feet/sec. (2) 2·513 sec.
(3) 2 646, 3·464, 3·873 cm./sec.
(4) 80, 75, 60, 35, 0 ergs. (5) i. 30; ii. 27·98; iii. 22·36.
(7) ·897 sec.

CHAPTER XIX. MECHANICS OF FLUIDS (pp. 351—354)

(1) i. 1·620 × 10^8; ii. 2·347 × 10^3; iii. 3·380 × 10^5; iv. 1·048.

(3) On each vertical face 3395 lb. wt.; on bottom, 6789 lb. wt.

(4) 1·0534 × 10^6 dynes per sq. cm.

(5) i. 76 cm.; ii. 1033·6 cm.; iii. 1007·4 cm.

(7) i. 1·0138 × 10^6; ii. 1033·3; iii. 14·69.

(9) 112,320 lb. wt. = 50·14 tons wt.; position, $12\frac{2}{9}$ feet from upper level of water.

(14) $h^2 r\rho g$.

(15) i. Vertical and equal to $\frac{2}{3}\pi r^3\rho g$; ii. $\frac{1}{3}\sqrt{\pi^2 + 4} \, . \, r^3\rho g$ making with horizontal $\tan^{-1}\frac{\pi}{2}$.

(16) $\frac{1}{2}\sqrt{\pi^2 + 16} \, . \, r^2 l\rho g$ making with horizontal $\tan^{-1}\frac{\pi}{4}$.

(17) 6·93 cub. inches. (18) ·0338. (19) 19·75 cm.

(20) $(\cdot 91)^2 = \cdot 8281$.

(22) $\frac{3}{4} h \left(\sqrt[3]{\frac{\rho_s}{\rho_l}\sec.^2\alpha} - 1 \right)$ where h = length of axis, α = semi-vertical angle, ρ_s = density of solid and ρ_l = density of liquid.

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